CALCULUS FOR SCHOOLS

FAWDRY AND DURELL

P. J. Morris

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PART II.—General Methods of Differentiation; Derivatives of Trigonometrical Functions; Logarithmic and Exponential Functions; Differentials and General Integration; Applications to Geometry; Complex Number and the Hyperbolic Functions; Expansions in Series. Table of Napierian Logarithms. Table for e^x, e^{-x}, $\cosh x$, $\sinh x$. Answers.

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PREFACE

THERE have been many changes in the teaching of elementary mathematics during the last twenty years, but none perhaps so significant as the growing tendency to regard any mathematical education as incomplete which does not include something of the fundamental ideas of the Calculus. This is a natural consequence of the belief that pupils towards the end of their time at School gain more from encountering new ideas than strengthening their power of technique in formal Algebra and Trigonometry. The authors have attempted in this volume to develop a systematic course which, while making small demands on the manipulative skill of the reader, calls for continual thought and makes progress depend rather on the exercise of common-sense than on the mechanical use of rules.

Those who have taught this subject to non-specialists will agree that the real initial difficulty is of the same nature as that which presents itself in approaching Analytical Geometry. From the plotting of a graph from a given equation to a real grasp of the meaning of "the equation of a curve" is a big step, and it smooths the path to spend time in driving home this idea before embarking on the ordinary graphical illustrations which lead up to the peculiar notation of the Calculus. For this reason the authors attach great importance to the subject matter of Chapter I which might at a first glance perhaps appear irrelevant.

The discussions in the text have been cut down to the smallest dimensions consistent with indicating the plan of procedure and establishing the necessary bookwork. The nature of the course depends much more on the character of the exercises than on the actual bookwork and so every effort has been made to select

examples which exhibit the practical applications of the subject and the variety of its ideas.

The only discussion of any length in Part II is that which introduces the logarithmic and exponential functions. The method adopted is designed to show the reader that, just as the handling of the trigonometric functions is simplified by the change of measurement from degrees to radians by the introduction of π, so the inconvenience of working with an awkward numerical factor which appears in the differentiation of 10^x and $\log_{10} x$ is avoided by changing from the base 10 to a new base e whose value may be obtained approximately by using Simpson's rule. This emphasis on practical convenience tends to remove the mystery which so often surrounds e and e^x in the pupil's mind. A table of Napierian logarithms is provided to facilitate calculation and make the work more concrete.

Hyperbolic functions are slowly coming into general use: they are no more difficult to handle than the trigonometric functions, and offer the best means of evaluating various types of integrals. On account of their practical importance, a chapter is devoted to them and a table of values of the function for use in examples is given at the end of the book.

The scope of the whole work is indicated by the Table of Contents but it may be useful to mention that Parts I and II together include all that is required for the Army Entrance examination and the Qualifying examination for the Mechanical Sciences Tripos at Cambridge.

Acknowledgment is due to the Controller of H.M. Stationery Office and to the Syndics of the Cambridge University Press and to the Oxford and Cambridge Joint Board for kind permission to include questions set in recent examinations.

C. V. D.
R. C. F.

April, 1923.

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CALCULUS FOR SCHOOLS

PART I

CHAPTER I

COORDINATE GEOMETRY

Coordinates

THE position of a point in a plane with reference to two rectangular axes may be fixed by stating the perpendicular distances of the point from the axes. It is convenient to draw only the perpendicular PN on the X-axis: the lengths ON (x) and NP (y) then fix the positions of P. x and y are known as the *coordinates* of P; x is called the *abscissa* and y the *ordinate*. For points such as P_2, P_3, P_4 the usual convention of signs is adopted and it should be noted that the abscissa is always written first; thus the coordinates of P_2 are $(-2, 1)$, of P_3 $(-3, -2)$ and of P_4 $(4, -1)$.

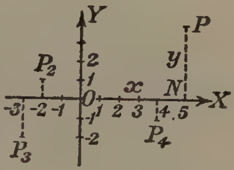

Fig. 1

Equations of Curves

The student is probably familiar with the drawing of a graph of such an equation as $y = x + 4$, but it is essential to have a clear idea of what this operation means.

We take any values of x (called the independent variable) and calculate the corresponding values of y (called the dependent variable, since its value depends upon the value taken for x). We have

x	-3	-2	-1	0	1	2
y	1	2	3	4	5	6

Each pair of values of x and y (e.g. $x=-3$, $y=1$) is now plotted and a number of points are obtained which in this case all lie on a straight line. The equation $y=x+4$ is called the equation of this line because *the coordinates of any point whatever on this line are connected by the relation*

$$y=x+4.$$

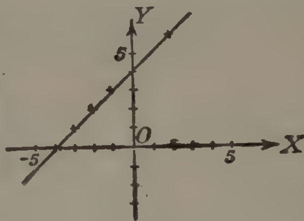

Fig. 2.

Similarly for the relation $2y=x^2$ we have

x	-2	-1	0	1	2	3
y	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$

The points $(-2, 2)$, $(-1, \frac{1}{2})$, $(0, 0)$, etc. lie on a curve (a parabola) whose equation is $2y=x^2$ and all points whose coordinates satisfy the relation $2y=x^2$ will lie on this curve. Further, if any point whatever is taken on the curve and if the x-distance and y-distance of this point are measured, the values of x and y so obtained must satisfy the equation $2y=x^2$.

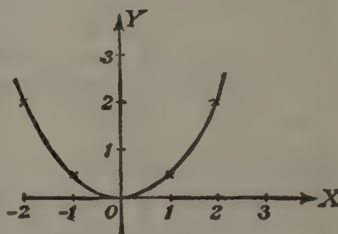

Fig. 3.

Hence we see that *the equation of a curve is the relation between the coordinates of all points on the curve.*

A very important aspect of a graph is to look upon it as the locus of a moving point tracing out a curve in a manner similar to the curve traced out on a rotating cylinder by a pen recording the varying barometric pressure. For the curve $2y=x^2$ we see for large negative values of x that y is large and positive, then as x approaches 0, y diminishes and also approaches 0, and when x

increases positively, y also increases positively and more rapidly than x.

For convenience we always travel from negative to positive values of x, thus making positive increments in x. If the curve is descending, the corresponding change in y is negative; if the curve ascends, the increment in y is positive.

EXAMPLES I a

1. In Fig. 4 the equation of the line $ABCD$ is $3x-5y+15=0$:

(i) If $ON_1=1$, what is P_1N_1?

(ii) If $P_1N_1=3$, what is ON_1?

(iii) If $ON_2=-2$, what is P_2N_2?

(iv) If $P_3N_3=-6$, what is ON_3?

(v) What are OC and OB?

(vi) Which of the following points lie on AD, produced if necessary: (5, 6); (−4, 1); (−7, −1·2)?

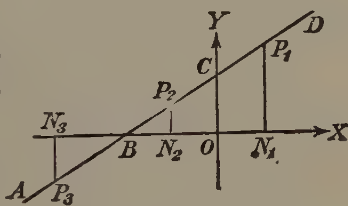

Fig. 4.

(vii) If $ON_1=2t$, what is P_1N_1?

(viii) If $P_1N_1=a+3$, what is ON_1?

2. In Fig. 5 the equation of the line $ABCD$ is $4x+5y=10$:

(i) If $ON=1$, what is PN?

(ii) If $QM=-2$, what is OM?

(iii) What are OC, OB?

(iv) Which of the following points lie on AD: (1, 1·2); (−5, 6); (9, −5)?

(v) If $ON=a$, what is PN?

(vi) If $ON=x+h$, what is PN?

(vii) If $PN=2t$, what is ON?

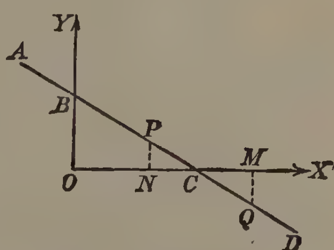

Fig. 5.

3. In Fig. 6 $OA=3$, $ON=2$, $PN=4{\cdot}5$, $OM=x$, $QM=y$:

(i) Find an equation between x and y.

(ii) What is OB?

(iii) Find the angle PBN.

(iv) Does (8, 9) lie on the line AB?

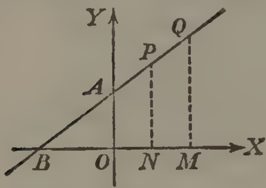

Fig. 6.

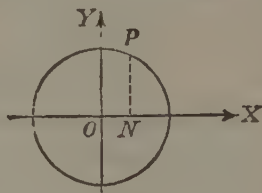

Fig. 7.

4. Fig. 7 represents a circle centre O, radius 5. $ON=x$, $PN=y$:

(i) What equation connects x and y?

(ii) Does the point (3, 4) lie on the circle?

(iii) If $ON=2a$, find PN in terms of a.

5. The equation of the curve in Fig. 8 is $y=\frac{1}{4}x^2$:

(i) If $ON=4$, $NM=2$, find PN, QM, QR and $Q\hat{P}R$.

(ii) If $ON=a$, $NM=h$, find PN, QM, QR, $\dfrac{QR}{PR}$.

(iii) If $PN=2{\cdot}25$, what is ON?

(iv) Does (10, 25) lie on the curve?

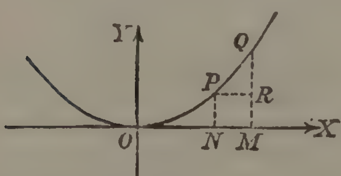

Fig. 8.

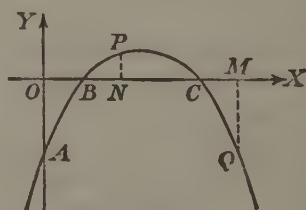

Fig. 9.

6. The equation of the curve shown in Fig. 9 is $y=(x-1)(4-x)$:

(i) If $ON=2$, what is PN?

(ii) If $OM=5$, what is QM?

(iii) What are OB, OC, OA?

(iv) If $QM=-10$, what is OM?

(v) Which of the following points lie on the curve:

$$(3, 2);\ (10, -55);\ (-2, -18)?$$

(vi) If $ON=a$, what is PN?

(vii) If $ON=a+h$, what is PN?

7. Draw freehand smooth curves whose equations satisfy the conditions in the following tables:

(i)

x = large negative	-2	0	small positive	5	large positive
y = large negative	0	0	small negative	0	large positive

(ii)

x = large negative	small negative	small positive	large positive
y = large positive	small negative	small positive	large negative

(iii) x is never greater than 2;

y is never greater than 3;

the curve is symmetrical about each axis.

8. Fig. 10 represents the curve whose equation is $y=3x-x^2$. Copy this freehand and then, with the same unit and axes, sketch the curves whose equations are

(i) $y=1+3x-x^2$;

(ii) $y=3(3x-x^2)$;

(iii) $y=x^2-3x$;

(iv) $y=3x-x^2-2$.

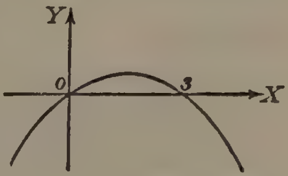

Fig. 10.

9. A curve is drawn whose equation is $y=x^2+3$:

(i) Does any part of it lie below the axis Ox?

(ii) What can you say about the value of y if x has (*a*) a large negative value, (*b*) a large positive value?

(iii) What is the least value y can have?

(iv) If $y=12$, what can you say about the value of x?

(v) Sketch the shape of the curve.

10. A curve is drawn whose equation is $y=x^3$:

(i) Which of the points $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$ lie on it?

(ii) Does any part of it lie below the axis Ox?

(iii) What can you say about the value of y if x has (*a*) a large negative value, (*b*) a large positive value?

(iv) If $(a, 8)$ lies on it, what is a?

(v) Sketch the shape of the curve.

11. The curve in Fig. 11 represents a certain equation which expresses y in terms of x:

(i) Describe in general terms how y changes as x increases from large negative to large positive values.

(ii) The scale is shown on the figure, state approximately the value of y for $x=-3$, -1, $1{\cdot}5$, 5; and the value of x for $y=-2$, 1, 5.

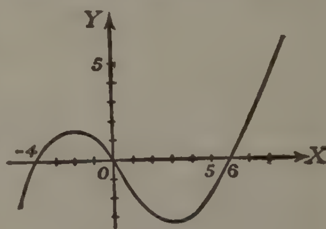

Fig. 11.

Functions

The formula $A=\pi r^2$ for the area of a circle is a statement of the fact that the area of a circle varies as the square of the radius. If we alter r, we alter the value of A. Here A is said to be a *function* of r; π remains the same for all circles and is called a *constant.*

Similarly the formula $t = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ expresses the fact that the time of swing of a pendulum is a function of its length, $\frac{2\pi}{\sqrt{g}}$ being a constant (if l is measured in feet and t in seconds the value of this constant is approximately 1·11).

Again the formula $S = \pi r l$ shows that the area S of the curved surface of a cone is a function of r, the radius of the base, and l, the slant height.

When variables are so related that the value of one of them depends upon the values of the others, the first variable is said to be a function of the others.

When we are studying laws for the treatment of a class of variables (e.g. laws which govern the behaviour of y when x varies, y being connected with x by some relation such as $y = x^2$ or $y = x^3 + \frac{3}{4}x^2 - x + 2$), we use the symbol $y = f(x)$ which is read $y =$ function x.

If we wish to distinguish between the two functions given above we could write $y = f(x)$ for the first and $y = F(x)$ or $y = \phi(x)$ or $y = f_1(x)$ for the second.

The result obtained by putting a instead of x in the expression represented by $f(x)$ is written $f(a)$.

Thus if $y = f(x) \equiv x^2$, then $f(a)$ would represent a^2 and $f(2b+3)$ would represent $(2b+3)^2$.

Again if $$y = \phi(x) \equiv x^3 + \tfrac{3}{4}x^2 - x + 2,$$
then $\phi(z)$ would stand for
$$z^3 + \tfrac{3}{4}z^2 - z + 2$$
and $\phi(1-y)$ would stand for
$$(1-y)^3 + \tfrac{3}{4}(1-y)^2 - (1-y) + 2.$$

Explicit and Implicit Functions

When y is expressed directly in terms of x, as in the examples given above, it is said to be an *explicit function* of x.

When the relation between x and y is given in a form such as $x^2 + y^2 = a^2$ (a circle), y is said to be an *implicit function* of x.

If, however, we write this result in the form $y=\pm\sqrt{(a^2-x^2)}$, y is expressed as an explicit function of x.

Of course we can alternatively take y as the independent variable and say that x is a function of y: but, unless otherwise stated, it will always be assumed that the independent variable is represented by x.

EXAMPLES I b

1. If $f(x)\equiv x^3-3$, find the values of

$$f(1),\ f(-1),\ f(0),\ f(3)-f(2),\ f(2y),\ f(a+b).$$

2. If $\phi(x)\equiv x^2-x$, find the values of

$$\phi(0),\ \phi(1),\ \phi(\tfrac{1}{2}),\ \phi(3t),\ \phi(x^2),\ \phi(x+1),\ \phi(-x).$$

3. If $f(x)\equiv x^2-2x+3$, find the values of $f(1+a)$, $f(x+a)$, $f\left(\frac{1}{x}\right)$.

4. If $\phi(x)\equiv 10^x$, find $\phi(1)$, $\phi(-1)$.

5. If $F(x)\equiv\dfrac{x^2+x+1}{x}$ prove that $F(x)=F\left(\dfrac{1}{x}\right)$.

6. If $f(x)\equiv x^2-2x$, find $\dfrac{f(x+h)-f(x)}{h}$. What is the approximate value of your result when h is small compared with x?

7. Find the values of x which make $f(x)=0$, given that

$$f(x)\equiv(x-2)(x+3).$$

8. If $f(x)\equiv\sin x$, find $f(0)$, $f\left(\dfrac{\pi}{2}\right)$.

9. If $\phi(x)\equiv\dfrac{1}{x}$, find $\dfrac{\phi(x+h)-\phi(x)}{h}$. What is the approximate value of your result when h is small compared with x?

10. If $f(x)=x^2+5$, what is the increase in $f(x)$ when x increases from (i) 1 to 2, (ii) 2 to 3?

11. Does $f(x)$ increase more or less rapidly in the interval $x=1$ to $x=2$ than in the interval $x=3$ to $x=4$ if

$$\text{(i) } f(x)\equiv x^3,\quad \text{(ii) } f(x)\equiv 10x-x^2\,?$$

12. If $f(x)=\dfrac{24}{x}$, what is the *increase* in $f(x)$ when x increases from (i) 2 to 3, (ii) 3 to 4?

Rough graphs

It is often desirable to obtain the shape and position of a curve by drawing a rough graph; the following examples will indicate the points to be observed:

1. Plot the curve $y=x^2$.

Notes.

(*a*) Since x^2 is positive, no negative values of y are possible.

(*b*) For any value of y there are two equal values of x of opposite sign, in other words the equation is unaltered if we substitute $-x$ for x. The curve is therefore symmetrical about OY.

(*c*) The equation is satisfied by (0, 0). The curve therefore goes through the origin, and its shape is shown in Fig. 12.

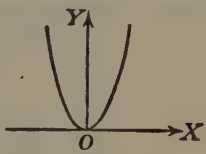

Fig. 12.

2. Plot the curve $y=(x-2)(x+1)(x-3)$.

Notes.

(*a*) $y=0$ when $x=2$, -1, and 3.

(*b*) When x is large positive, y is large positive.
When x is large negative, y is large negative.

(*c*) When $x=0$, $y=6$.

(*d*) When $3>x>2$, i.e. when x lies between $+2$ and $+3$, y is negative.
When $2>x>-1$, y is positive.

Fig. 13 represents the required curve.

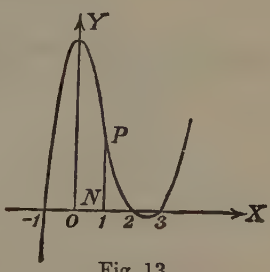

Fig. 13.

3. Plot the curve $xy-2y-4=0$.

Notes.

(*a*) Solving for y we have $y=\dfrac{4}{x-2}$.
When $x=0$, $y=-2$.
When x is large positive, y is small positive.
When x is large negative, y is small negative.

When $x \to 2$, y becomes very large.
When x is slightly greater than 2, y is positive.
When x is slightly less than 2, y is negative.

(*b*) Solving for x we have $x=2+\frac{4}{y}$.

When y is large, $x=2+\frac{4}{y}$ will differ very slightly from 2 and the curve will therefore continually approach the position of the line $x=2$ as y increases. Such a line is called an asymptote of the curve.
When $y \to 0$, x becomes very large and the curve approaches the line $y=0$, which is also an asymptote.
Fig. 14 represents the required curve.

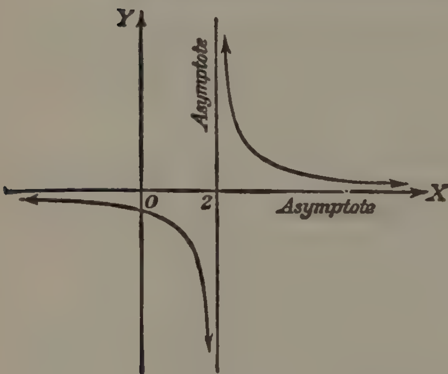

Fig. 14.

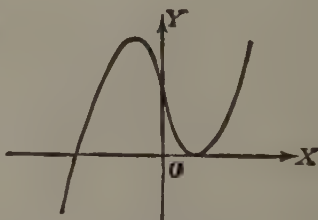

Fig. 15.

4. Plot the curve $y=(x-1)^2(x+3)$.

Note here that if the equation were $y=(x-1{\cdot}1)(x-{\cdot}9)(x+3)$ y would be zero when $x=1{\cdot}1$, $\cdot 9$ and -3; the curve would therefore cut OX at two points $x=1{\cdot}1$ and $x={\cdot}9$ very close together.
The curve $y=(x-1)(x-1)(x+3)$ therefore touches the x-axis at $x=1$: it also cuts it again at $x=-3$.
Further $y>0$ if $x>-3$ and $y<0$ if $x<-3$.
The form of the curve is represented by Fig. 15.

EXAMPLES I c

1. State between what values of x the following expressions are (1) positive, (2) negative:

$$(x-1)(3+x);\ (x-2)^2(x^2-9);\ (x+2)(\tfrac{3}{2}-x).$$

2. Draw rough graphs of

(1) $y=2x+\dfrac{2}{x}$; (2) $y=x^3$; (3) $y=x^2+\dfrac{1}{x}$;

(4) $y^2-4x+16=0$; (5) $x^2y=(x-1)^2$; (6) $xy-3x-y+4=0$;

(7) $xy^2=4(3-x)$; (8) $y=\dfrac{x^2}{1+x^2}$.

3. Draw with the same axes rough graphs of $y=x^n$ when

$$n=1,\ 2,\ 3,\ 4,\ \dots,\ -1,\ -2,\ -3,\ \dots,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \dots.$$

4. Draw the graph of

$$y=(x-2)(x+1)(x+2)$$

and from it deduce the graph of

$$y=\sqrt{(x-2)(x+1)(x+2)}.$$

5. Sketch the graph of $y=f(x)$ if $f(x)\equiv 2x+1$.

6. Sketch the graph of $y=f(x)$ if $f(x)\equiv x^2-1$.

7. Fig. 16 represents the graph of

$$y=f(x).$$

Express PN in the form $f(x)$ if

(i) $ON=3$, (ii) $ON=a$,
(iii) $ON=x$, (iv) $ON=x+h$,

how would you represent OC?

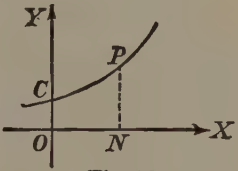

Fig. 16.

8. Fig. 8, p. 4, represents the graph of $y=f(x)$. Express in functional notation QR if $ON=x$, $NM=h$.

9. Fig. 17 represents the graph of $y=\phi(x)$. Copy it freehand and draw lines to represent

(i) $\phi(-2)$, (ii) $\phi(0)$,
(iii) $\phi(2)$, (iv) $\phi(3{\cdot}5)$.

What can you say about

$\phi(-1)$, $\phi(3)$, $\phi(4)$?

Which is the larger, $\phi(1)$ or $\phi(2)$?

Sketch the graph of $y=2\phi(x)$.

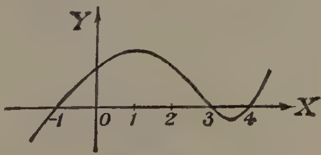

Fig. 17.

10. Fig. 16 represents the graph of $y=f(x)$. What lines can you draw to represent

$$\text{(i) } f(a), \quad \text{(ii) } f(a+h), \quad \text{(iii) } f(a-h)?$$

If
$$\tan\theta = \frac{f(a+h)-f(a)}{h},$$

what angle in your figure is equal to θ?

11. Sketch a smooth curve whose equation is $y=f(x)$ for the following cases:

(i) If x is positive, $f(x)$ is positive and increases as x increases;

(ii) If x is positive, $f(x)$ is negative and increases as x increases;

(iii) If x is positive, $f(x)$ is positive and decreases as x increases.

12. Fig. 18 represents the graph of $y=f(x)$ for values of x from 0 to 4. If

$$f(x)=f(x+4n)$$

for all positive and negative integral values of n and all values of x, what does the rest of the graph look like?

What are the values of $f(1)$, $f(3)$, $f(19)$, $f(25)$, $f(40)$, $f(-7)$?

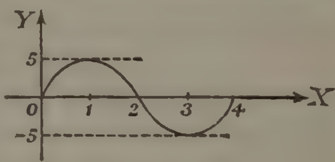

Fig. 18.

13. Sketch a curve whose equation is $y=f(x)$ where, for positive values of x, $f(x)$ increases as x increases, but the rate of increase of $f(x)$ decreases as x increases.

14. Sketch the smooth curve whose equation is $y=f(x)$ for values of x from 0 to 10, given that

$x=0$	0 to 2	2 to 6	6 to 8	8 to 10	10
$f(x)=0$	increasing	decreasing and positive	decreasing and negative	increasing and negative	0

CHAPTER II

GRADIENTS

When we say that the gradient of a road is 1 in 10, we mean that the road rises 1 unit for every 10 units we travel on it. Trigonometrically, $\frac{1}{10}$ = the sine of the slope (α) of the road, i.e. $\sin \alpha = \frac{1}{10}$.

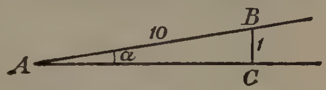

Fig. 19.

In Geometry, we measure the gradient of the line AB by the ratio $\frac{CB}{AC}$, that is by $\tan \alpha$, provided that CB is drawn on the same scale as AC. For ordinary roads, the difference between $\tan \alpha$ and $\sin \alpha$ is negligible; thus in the present case $\sin 5° 44' = 0{\cdot}1$ and $\tan 5° 43' = 0{\cdot}1$.

Since $\frac{BC}{AB}$ is approximately the radian measure of α when α is small, we see that by taking a radian as 60° (approximately) we obtain a simple rough rule for expressing gradients as angles. Thus a gradient $\frac{1}{10}$ corresponds to an angle of about $\frac{1}{10} \times 60°$ or 6°.

Gradient of a straight line

When we travel along the straight line AB, starting at P where $ON = x$, $NP = y$, an increment $NM = PR$ in x produces the increment $MQ - NP = RQ$ in y.

$\therefore$ a unit increase in x produces the increase $\frac{RQ}{PR}$ in y.

The ratio $\frac{RQ}{PR}$ is called the *gradient* of the line AB. If we had taken an increment $NM_1 = PR_1$ in x, the resulting increment in y would have been

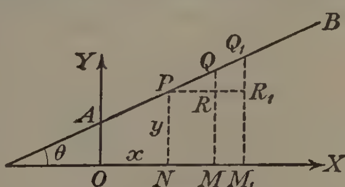

Fig. 20.

$M_1Q_1 - NP = R_1Q_1$. And then the gradient would be $\frac{R_1Q_1}{PR_1}$. But by similar triangles $\frac{R_1Q_1}{PR_1} = \frac{RQ}{PR}$.

∴ *the gradient of a straight line* does not depend on the size of the increment taken.

Suppose the line AB makes an angle θ with the *positive direction* of Ox, then $\angle QPR = \theta$, ∴ $\frac{RQ}{PR} = \tan\theta$, provided that the units on both axes are the same.

∴ the gradient of the line AB is $\tan\theta$ and its value does not depend on the position of the starting-point P.

The δx, δy Notation

The symbol δx is used to represent any increase in the value of x. The δ and the x cannot be separated, the symbol does *not* mean $\delta \times x$; it is simply a shorthand way of writing the phrase "any increase in x."

If y is given in terms of x, a change in the value of x causes in general a change in the value of y. The symbol δy is used to represent the increase in the value of y caused by the increase δx in the value of x. Any values can be given to x and δx, but as soon as these values are assigned, the values of y and δy can be calculated.

If y decreases as x increases, then δy is negative when δx is positive.

If in Fig. 20 we take $\delta x = NM$, then δy equals the corresponding increase in $y = MQ - NP = RQ$.

∴ the gradient $= \tan\theta = \frac{RQ}{PR} = \frac{\delta y}{\delta x}$.

If in Fig. 21 the point P, coordinates (x, y), lies on the line AB, we have $ON = x$, $NP = y$. Suppose $NM = \delta x$, then $\delta y =$ increase in the corresponding value of $y = MQ - NP = -QR$. [δy is negative because y becomes smaller as x becomes larger.]

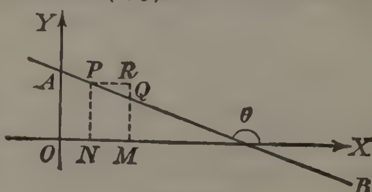

Fig. 21.

The gradient of AB is $\frac{\delta y}{\delta x}$ and this is negative and is still measured by $\tan\theta$, the tangent of an obtuse angle.

Note. It is convenient to use the phrase "the point P (x, y)" as meaning "the point P whose coordinates are (x, y)."

Gradient of a Curve

If in Fig. 22 the point P (x, y) lies on a given curve, we have $ON=x$, $NP=y$. Suppose $NM=\delta x$, then the corresponding increment in y is $RQ=\delta y$.

$$\therefore \frac{\delta y}{\delta x}=\frac{RQ}{PR}.$$

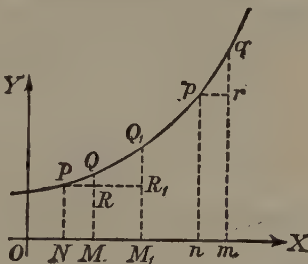

Fig. 22.

But if $\delta x = NM_1$, then $\delta y = R_1Q_1$,

and $$\frac{\delta y}{\delta x}=\frac{R_1Q_1}{PR_1}.$$

But $\frac{R_1Q_1}{PR_1}$ is not equal to $\frac{RQ}{PR}$.

$\therefore$ the value of $\frac{\delta y}{\delta x}$ depends on the magnitude of δx.

Again, suppose the point (x, y) is p, then as before $\frac{\delta y}{\delta x}=\frac{rq}{pr}$ which is not equal to $\frac{RQ}{PR}$.

$\therefore$ the value of $\frac{\delta y}{\delta x}$ depends also on the magnitude of x.

Consider the curve $y=x^2$ (see Fig. 8, p. 4). Suppose P is the point $(2, 4)$. Then $ON=x=2$, $NP=y=2^2=4$. If $\delta x=NM=0{\cdot}1$, $OM=2{\cdot}1$.

$$\therefore MQ=OM^2=(2{\cdot}1)^2=4{\cdot}41,$$
$$\therefore \delta y=MQ-NP=4{\cdot}41-4=0{\cdot}41,$$
$$\therefore \frac{\delta y}{\delta x}=\frac{0{\cdot}41}{0{\cdot}1}=4{\cdot}1.$$

In the same way we could calculate the value of $\frac{\delta y}{\delta x}$ for any other

point (x, y) and any other increment δx. In general, the result will be different.

In this case $\frac{\delta y}{\delta x}$ is called the *average gradient* over the interval NM, and it represents the gradient of the chord PQ.

EXAMPLES II a

1. If the equation of the line AB in Fig. 20, p. 13, is $3x-5y+12=0$, calculate QR, Q_1R_1, $\frac{QR}{PR}$, $\frac{Q_1R_1}{PR_1}$, given (i) $ON=3$, $NM=1$, $NM_1=1{\cdot}5$, (ii) $ON=a$, $NM=h$, $NM_1=k$. Find also the value of θ.

2. If the equation of the line AB in Fig. 21, p. 14, is $3x+5y=11$, calculate $\frac{QM-PN}{NM}$, given (i) $ON=2$, $NM=0{\cdot}5$, (ii) $ON=a$, $NM=h$. Find also the value of θ.

3. Calculate the gradients of the lines whose equations are

(i) $y=2x-5$, (ii) $x=3y+1$.

4. If $y=8-5x$, find δy if (i) $x=3$, $\delta x=0{\cdot}1$ or (ii) $x=a$, $\delta x=0{\cdot}1$.

5. If $y=3x-4$, find δy in terms of δx. What is the value of $\frac{\delta y}{\delta x}$? Explain by a figure why the value of $\frac{\delta y}{\delta x}$ does not depend on the value of x.

6. If the equation of the curve in Fig. 22, p. 15, is $y=x^2+x+2$, calculate QR, Q_1R_1, $\frac{QR}{PR}$, $\frac{Q_1R_1}{PR_1}$, given (i) $ON=1$, $NM=1$, $NM_1=2$, (ii) $ON=a$, $NM=h$, $NM_1=k$. Find δy and $\frac{\delta y}{\delta x}$ in terms of x and δx.

7. Fig 10, p. 5, represents the graph of $y=3x-x^2$. What is $\frac{\delta y}{\delta x}$ if (i) $x=1$, $\delta x=1$, (ii) $x=1$, $\delta x=0{\cdot}1$, (iii) $x=1$, $\delta x=0{\cdot}001$, (iv) $x=2$, $\delta x=1$, (v) $x=2$, $\delta x=0{\cdot}1$, (vi) $x=2$, $\delta x=0{\cdot}001$?

Find $\frac{\delta y}{\delta x}$ in terms of x and δx.

8. If $y=f(x)$ is the equation of the curve in Fig. 22, p. 15, and if $ON=x$, $NM=\delta x$, how do you represent PN and QM in terms of x, δx?

What is the geometrical meaning of $\frac{f(x+\delta x)-f(x)}{\delta x}$?

9. Make a rough drawing to show a straight line of gradient (i) 2, (ii) $\frac{1}{2}$, (iii) -2, (iv) 0.

10. Fig. 23 represents a semi-circle of radius 2″, centre C; M, N are the mid-points of OC, CA; PM, QN are ordinates. Calculate the average gradients of the circle over the intervals OM, MC, CN, NA.

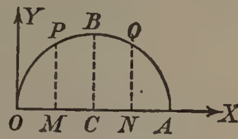

Fig. 23.

11. A square of side x ins. is of area A sq. ins.

(i) Express A in terms of x.

(ii) Express δA in terms of x, δx and illustrate your answer by a figure.

12. A stone is dropped and falls s feet in t seconds; if $s=16t^2$, express δs in terms of t, δt. By substituting in this result, find the distance it goes in the interval $t=2$ to $t=2{\cdot}1$. What is its average speed over this interval?

13. A certain mass of gas when under a pressure of p lbs. per sq. in. occupies v cu. ins. where $pv=500$. Express (i) v in terms of p, (ii) δv in terms of p and δp. What is the change in volume when the pressure is increased from 50 to 55 lbs. per sq. in.?

14. If a 5″ shell is travelling at v feet per second, the air resistance is R lbs. where $R=\frac{1}{10^7}\cdot v^3$; (i) express δR in terms of δv; (ii) find the alteration in resistance when the speed falls from 1000 to 900 feet per second.

15. The radius of a circle being x, show that a small change δx in x will result in a change of $2\pi x\,\delta x$ approximately in the area. Hence find the approximate area between two concentric circles of radii 2″ and 2·1″.

16. The volume of a sphere being $\frac{4}{3}\pi r^3$, find the volume of the outer cover of a cricket ball whose outside radius is 10 cms. and thickness of cover 3 mms. (Neglect $(\delta r)^2$ and $(\delta r)^3$.)

Limits

In Fig. 24, P is the point (2, 4) on the curve $y=x^2$. It was shown on p. 15 that for $x=2$, $\delta x=0{\cdot}1$, we have $y=4$, $\delta y=0{\cdot}41$ and $\frac{\delta y}{\delta x}=4{\cdot}1$.

Similarly if $\delta x=0{\cdot}01$, $OM=2{\cdot}01$.

$\therefore\ y+\delta y=(2{\cdot}01)^2=4{\cdot}0401.$

$\therefore\ \delta y=0{\cdot}0401$ and $\frac{\delta y}{\delta x}=\frac{0{\cdot}0401}{0{\cdot}01}=4{\cdot}01.$

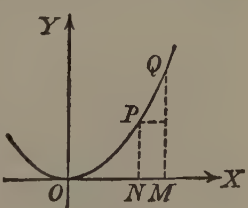

Fig. 24.

By similar calculations, we can obtain the results given in the following table.

When $x=2$, $y=4$:

$\delta x = 0{\cdot}1$	$0{\cdot}01$	$0{\cdot}001$	$0{\cdot}0001$
$\delta y = 0{\cdot}41$	$0{\cdot}0401$	$0{\cdot}004001$	$0{\cdot}00040001$
$\dfrac{\delta y}{\delta x} = 4{\cdot}1$	$4{\cdot}01$	$4{\cdot}001$	$4{\cdot}0001$

It is evident that the smaller δx becomes, the nearer $\dfrac{\delta y}{\delta x}$ approaches 4: but it is impossible to choose a value of δx which will make $\dfrac{\delta y}{\delta x}$ exactly equal to 4. This value 4 to which the gradient of the chord PQ can be made to approach as closely as we please but yet never quite reach is called its *Limiting value.*

The *gradient of the curve* at P is defined as the limit to which the gradient of the chord PQ continually approaches as Q approaches P.

Although the gradient of the chord depends on the magnitude of δx, the value of its *limit* does not depend on δx. At P (2, 4) on the curve $y=x^2$, the gradient of the curve is 4.

The idea of a limit is of fundamental importance in the Calculus and is illustrated in the following examples.

Example 1.

Consider the G.P. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$ to ∞. If we represent unity by a length AB, the first term will be represented by AB. If $BC=1$ and is bisected at D, $BD=\frac{1}{2}$ will represent the second term. Bisect DC at E, then DE represents the third term. By continually bisecting what is left we obtain successive terms of the series, but by this process we can never reach C although we can approach as near as we please to it: and the more terms we take the closer their sum approaches to 2: and the *limit* of the sum of the series is 2.

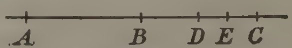

Fig. 25.

Example 2.

If we join A to a series of points B_1, B_2, B_3,... on a straight line, the farther we travel along the line from B_1 the more nearly does AB approach the position of a parallel through A to the line: and the parallel is the limiting position of AB_n as B_n moves to infinity.

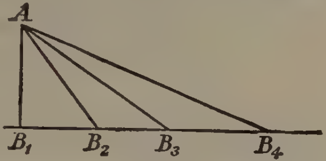

Fig. 26.

Example 3.

Consider the fraction $\dfrac{x^2-4}{x-2}$ when x approaches 2. At $x=2$, the fraction assumes the form $\frac{0}{0}$ which has no meaning. Let $x=2+h$, then the fraction$=\dfrac{(2+h)^2-4}{2+h-2}=\dfrac{4h+h^2}{h}=4+h$ if $h\neq 0$. As h approaches zero, x approaches 2 and the fraction approaches the value 4. By taking x sufficiently near 2, we can make the fraction take a value as near 4 as we like: and the nearer x becomes to 2, the nearer the fraction becomes to 4. We say that the *limit* of $\dfrac{x^2-4}{x-2}$ *equals* 4 when x tends to 2, in spite of the fact that actually for $x=2$ the fraction has no meaning.

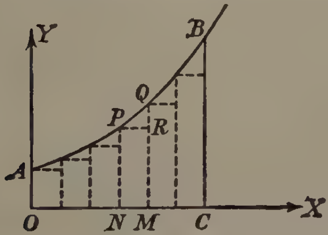

Fig. 27.

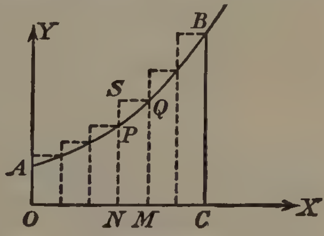

Fig. 28.

Example 4.

Suppose OC is divided into any number of equal parts of which NM is one, and rectangles are then drawn as shown in Figs. 27, 28.

The sum of the areas of the rectangles in Fig. 27 is less than the area of the curve $ABCO$ which is less than the sum of the areas of the rectangles in Fig. 28.

Suppose OC is divided into n equal parts, then the difference between the sum of the rectangles in Fig. 27 and the sum of the rectangles in Fig. 28 is $\dfrac{OC}{n}(CB-OA)$, for each rectangle is of breadth $\dfrac{OC}{n}$.

By taking n large enough, this difference can be made as small as we please.

Hence the limit of the sum of the areas of the rectangles in Fig. 27 when $n \to \infty$ is equal to the limit of the sum of the areas of the rectangles in Fig. 28 and we call this limit the area of the curve $ABCO$.

Example 5.

If the angle AOP is θ radians, where O is the centre of the circle PAQ, then $\theta = \dfrac{\text{arc } PA}{OP}$.

PN is perpendicular to OA and cuts the circle again at Q; the tangents at P, Q meet at T on OA produced.

We have $\tan\theta = \dfrac{PT}{OP}$, $\sin\theta = \dfrac{PN}{OP}$.

Now $\quad TP + TQ > \text{arc } PAQ > PNQ.$

$\therefore PT > \text{arc } PA > PN.$

$\therefore OP\tan\theta > OP.\theta > OP\sin\theta.$

Divide through by $OP\sin\theta$,

$$\therefore \frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1.$$

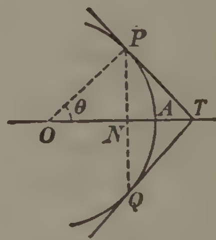

Fig. 29.

But as θ approaches zero, $\cos\theta \to 1$.

$$\therefore \underset{\theta\to 0}{\text{Lt}} \frac{\theta}{\sin\theta} = 1.$$

Note. It is important to realise that if two quantities each tend to zero together, it is unusual for their ratio to approach the value 1.

For example when $x \to 0$, the quantities $2x$ and $3x$ each tend to zero, but their ratio is always $\frac{2}{3}$ as long as $x \neq 0$, i.e. $\underset{x\to 0}{\text{Lt}} \dfrac{2x}{3x} = \frac{2}{3}$, but $\dfrac{2x}{3x}$ has no meaning when $x = 0$.

Or again, let PON be a right-angled triangle such that $PN = 2ON$ and imagine P to move along PO towards O; then as P tends to O, the lengths of PN and ON each tend to zero, but their ratio remains equal to 2 however near P is to O. When P actually coincides with O, the ratio is meaningless.

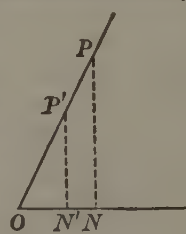

Fig. 30.

Example 6.

Find the limiting value of $\frac{n}{2n+1}$ as n tends to infinity.

Now
$$\frac{n}{2n+1}=\frac{1}{2+\frac{1}{n}}.$$

But the larger n becomes, the nearer does $\frac{1}{n}$ approach the value zero.

$$\therefore \mathop{\mathrm{Lt}}_{n\to\infty}\frac{n}{2n+1}=\mathop{\mathrm{Lt}}_{\frac{1}{n}\to 0}\frac{1}{2+\frac{1}{n}}=\tfrac{1}{2}.$$

The statement that the limit equals $\frac{1}{2}$ implies that we can make $\frac{n}{2n+1}$ approach $\frac{1}{2}$ within any given degree of approximation by taking n large enough.

For example $\frac{n}{2n+1}$ would differ from $\frac{1}{2}$ by less than 0·000,001 if n has any value $>300{,}000$.

The method employed in Example 6 is of frequent use: it really involves a change of variable; instead of evaluating the limit of a function of n when $n\to\infty$, we evaluate the limit of a function of u where $u=\frac{1}{n}$ when $u\to 0$.

EXAMPLES II b

1. (i) Tabulate the values of $\frac{x+1}{x}$ when x equals 1, 5, 10, 100, 1000.

(ii) Can you find a value of x for which $\frac{x+1}{x}$ equals 1?

(iii) Can you find a value of x for which $\frac{x+1}{x}$ differs from 1 by less than 0·000,001?

(iv) What is the limit of $\frac{x+1}{x}$ when x tends to infinity?

2. (i) Tabulate the values of $\frac{x^2-1}{x-1}$ when x equals 5, 4, 3, 2, 1·5, 1·1, 1·001.

(ii) Can you find a value of x for which $\frac{x^2-1}{x-1}$ equals 2?

(iii) Can you find a value of x for which $\dfrac{x^2-1}{x-1}$ differs from 2 by less than 0·0005?

(iv) What is the limit of $\dfrac{x^2-1}{x-1}$ when x tends to 1?

3. Plot the graph of $\dfrac{3n-1}{n-1}$ from $n=2$ to $n=10$. For what value of n is the difference between $\dfrac{3n-1}{n-1}$ and 3 equal to 0·0001? Find the limit of $\dfrac{3n-1}{n-1}$ when n tends to infinity.

4. (i) Can you find the value of $\dfrac{x^2-4}{x^2-2x}$ when $x=2$?

(ii) Does $\dfrac{x^2-4}{x^2-2x}$ tend to a limit when x tends to 2?

5. (i) Can you find the value of $\dfrac{(3+h)^2-9}{h}$ when $h=0$?

(ii) Does $\dfrac{(3+h)^2-9}{h}$ tend to a limit when h tends to 0?

(iii) Draw freehand the graph of $y=x^2$, and use it to represent geometrically the fraction $\dfrac{(3+h)^2-3^2}{h}$.

6. Show that the point P (1, 2) is on the curve $y=2x^2$. Take $x=1$, $y=2$, and complete the table:

$\delta x=0{\cdot}1$	0·01	0·001
$\delta y=$		
$\dfrac{\delta y}{\delta x}=$		

What is the limit of $\dfrac{\delta y}{\delta x}$ when $x=1$ and $\delta x \to 0$?

7. Fig. 31 represents the graph of $y=5x^2$. Take $ON=x$, $NM=\delta x$, and express QR in terms of x, δx. If $\angle PKN=\theta$, express $\tan\theta$ in terms of x, δx. To what limit does $\tan\theta$ tend when $\delta x \to 0$? Interpret this geometrically.

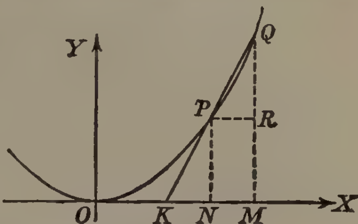

Fig. 31.

8. If $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ are points on the curve $y=x^3$, prove that

$$\delta y=3x^2 \,.\, \delta x+3x \,.\, (\delta x)^2+(\delta x)^3.$$

To what limit does $\dfrac{\delta y}{\delta x}$ tend when $\delta x \to 0$? Interpret this geometrically.

9. Show that the point P (2, 24) lies on the curve $y=3x^3$. Find the gradient of the chord joining the points on this curve whose abscissae are $2-h$ and $2+h$. To what limit does this result tend when $h \to 0$? Interpret this geometrically.

10. Draw a rough graph of $y=f(x)\equiv 3x-x^2$. Calculate the values of $f(a+h)$, $f(a)$, $\dfrac{f(a+h)-f(a)}{h}$ in terms of a, h. Represent these expressions geometrically. Evaluate $\underset{h\to 0}{\mathrm{Lt}}\dfrac{f(a+h)-f(a)}{h}$ and interpret this geometrically.

11. Draw a rough graph of $y=f(x)=\dfrac{12}{x}$.

Express $\dfrac{\delta y}{\delta x}$ in terms of x, δx and evaluate its limit when $\delta x \to 0$. Interpret this geometrically.

12. A stone is dropped and falls s feet in t seconds where $s=16t^2$. Prove that $\delta s=32t \,.\, \delta t+16 \,.\, (\delta t)^2$. What is the limit of $\dfrac{\delta s}{\delta t}$ when $\delta t \to 0$ and what does this mean?

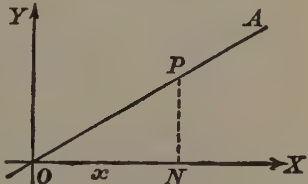

Fig. 32.

13. The equation of the line OA is $y=\frac{1}{2}x$. P is the point (x, y) and PN the ordinate. If the area of ΔONP is A units, express A in terms of x; hence find δA in terms of x, δx and interpret this geometrically.

What is the limit of $\frac{\delta A}{\delta x}$ when $\delta x \to 0$? Show that this limit is equal to PN and interpret geometrically.

14. A wine-glass contains v cu. ins. of wine when the depth of wine in it is x ins. where $v = \frac{1}{2}x^3$. Express δv in terms of x, δx and find the limit of $\frac{\delta v}{\delta x}$ when $\delta x \to 0$. Interpret this result.

15. (i) What is the average of the numbers 1, 2, 3, 4, ... 19, 20? What is their sum?

(ii) What is the average of the numbers 1, 2, 3, ... n? What is their sum?

(iii) Evaluate $\underset{n \to \infty}{\text{Lt}} \frac{1+2+3+\ldots+n}{n^2}$.

16. OP is the line $y = x$ from $x = 0$ to $x = 1$. Divide ON into n equal parts and construct n rectangles as in Fig. 33.

Find the sum of the areas of the n rectangles in terms of n. By using Ex. 15 (iii), find the limit of this sum when $n \to \infty$ and interpret this result geometrically.

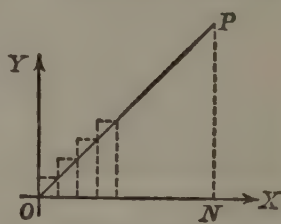

Fig. 33.

17. It can be proved that

$$1^2+2^2+3^2+4^2+\ldots+(n-1)^2 = \frac{n(n-1)(2n-1)}{6}.$$

(i) Use this result to evaluate $\underset{n \to \infty}{\text{Lt}} \frac{1^2+2^2+3^2+\ldots+(n-1)^2}{n^3}$.

(ii) OP is the curve $y = x^2$ from $x = 0$ to $x = 1$. Divide ON into n equal parts and construct $(n-1)$ rectangles as in Fig. 34. Find the sum of the areas of the $(n-1)$ rectangles in terms of n.

(iii) Find the limit of this sum when $n \to \infty$ and interpret this result geometrically.

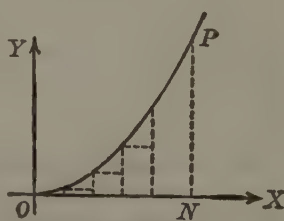

Fig. 34.

(iv) Prove that

$$1^2+2^2+3^2+4^2+\dots+(n-1)^2+n^2=\frac{n(n+1)(2n+1)}{6},$$

and evaluate $\underset{n\to\infty}{\mathrm{Lt}}\ \dfrac{1^2+2^2+\dots+n^2}{n^3}$.

(v) Draw a system of n rectangles overlapping the curve as in Fig. 28 and find the sum of their areas.

(vi) Find the limit of this sum when $n\to\infty$ and interpret this result geometrically and compare it with the result obtained in (iii).

18. Draw a curve to represent $y=f(x)$ and illustrate geometrically

$$\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(x)-f(x-h)}{h}.$$

19. Evaluate and interpret geometrically

(i) $\underset{\delta r\to 0}{\mathrm{Lt}}\ \dfrac{\pi(r+\delta r)^2-\pi r^2}{\delta r}$.

(ii) $\underset{\delta r\to 0}{\mathrm{Lt}}\ \dfrac{\frac{4}{3}\pi(r+\delta r)^3-\frac{4}{3}\pi r^3}{\delta r}$.

CHAPTER III

DIFFERENTIATION

The tangent to a curve

If we join the point P on a curve to another point Q on it, the chord PQ changes its position as Q approaches P: Fig. 35 represents successive positions PQ_1, PQ_2, PQ_3, which cut off arcs of diminishing lengths. Now it is possible to draw a tangent line PT in such a position that the chord PQ can be made to approach as nearly as we please the line PT but yet will never quite reach it. As long as P and Q are separate points on the curve, PQ is a chord and cannot coincide with the tangent PT. But the limiting position of the chord PQ when the length of the arc PQ tends to zero is the tangent PT. The position of the tangent cannot however be obtained by making Q coincide with P, for when Q is the same point as P, no definite position of a straight line can be obtained by joining them.

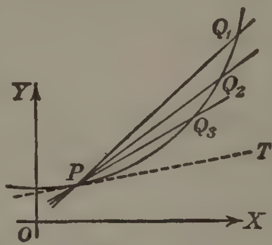

Fig. 35.

From the definition of the gradient of the curve at P (p. 18), it follows that the gradient of a curve at any point is the gradient of the tangent to the curve at that point.

Gradient of $y = x^2$ at $P(x, y)$

Let $P(x, y)$ be any point on the curve: take an increment $\delta x\ (= NM)$ in x and let $\delta y\ (= RQ)$ be the increment thus caused in y.

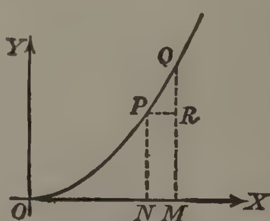

Fig. 36.

Then the coordinates of Q are

$$x + \delta x,\ y + \delta y.$$

Since Q lies on the curve, $MQ = OM^2$,

i.e. $\quad y + \delta y = (x + \delta x)^2 = x^2 + 2x(\delta x) + (\delta x)^2.$

Since P lies on the curve, $y = x^2$,

$$\therefore\ \delta y = 2x\,(\delta x) + (\delta x)^2.$$

$$\therefore\ \frac{\delta y}{\delta x} = 2x + \delta x.$$

This represents the gradient of the chord PQ and is called the average gradient of the curve over the interval NM.

The smaller δx becomes, the nearer $\frac{\delta y}{\delta x}$ approaches the value $2x$ but yet never reaches it. The gradient of the chord PQ, however small NM is, can never be equal to $2x$. But its limiting value is exactly $2x$. In symbols

$$\text{Lt}\ \frac{\delta y}{\delta x} = 2x \text{ when } \delta x \to 0.$$

$\therefore$ the gradient of the tangent to the curve at $P\,(x, y)$ or the gradient of the curve at P equals $2x$. This is an exact, *not* an approximate, statement.

The symbol used for $\underset{\delta x \to 0}{\text{Lt}}\ \frac{\delta y}{\delta x}$ is $\frac{dy}{dx}$.

In words, $\frac{dy}{dx}$ means the limiting value of the fraction $\frac{\delta y}{\delta x}$ as δx tends to zero.

The important fact to realise at this stage is that although δx and δy may be made as small as we please, it does not follow that the ratio $\frac{\delta y}{\delta x}$ becomes small. This fact was illustrated on p. 18.

Differential Coefficient

The limit $\frac{dy}{dx}$ is called the "differential coefficient of y with respect to x" or the "derivative of y with respect to x."

When y is a given function of x, the process of obtaining the limit $\frac{dy}{dx}$ is called "differentiating y with respect to x."

$\frac{dy}{dx}$ is sometimes written $\frac{d}{dx}(y)$. The symbol $\frac{d}{dx}$ is an operator, just in the same way that the symbol $\sqrt{}$ is an operator.

Thus $\sqrt{}\,(x^2+5x)$ means "take the square root of (x^2+5x)"; so $\frac{d}{dx}(x^2+5x)$ or $\frac{d(x^2+5x)}{dx}$ or $\frac{dy}{dx}$ where $y=x^2+5x$ means "find the limiting value of $\frac{\delta y}{\delta x}$ where $y=x^2+5x$ when $\delta x \to 0$." This operation is called "differentiation." The 'd' and the 'dx' cannot be separated any more than you could separate the dots and the line in the symbol $\div$ for the operation called "division."

Gradient of $y=3x^2$ at $P(x, y)$

Fig. 37 represents the graphs of $y=3x^2$ and $y=x^2$ drawn on the same scale.

$\therefore\ QM=3Q'M$ and $PN=3P'N$,

$\therefore\ QR=QM-PN$

$=3Q'M-3P'N=3(Q'M-P'N)=3Q'R'$,

$$\therefore\ \frac{RQ}{PR}=3\frac{R'Q'}{P'R'}.$$

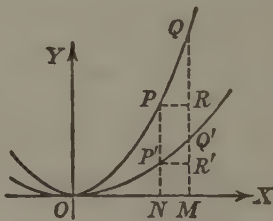

Fig. 37.

This relation holds good, however small NM is.

$\therefore$ in the limit, the gradient at $P=$ three times the gradient at P'.

$$\therefore\ \frac{d}{dx}(3x^2)=3\frac{d}{dx}(x^2)=3\times 2x=6x.$$

This result may be proved from first principles, as follows:

$$y=3x^2,$$

$$\therefore\ y+\delta y=3(x+\delta x)^2=3x^2+6x\,.\,\delta x+3\,.\,(\delta x)^2,$$

$$\therefore\quad \delta y=6x\,.\,\delta x+3\,.\,(\delta x)^2,$$

$$\therefore\quad \frac{\delta y}{\delta x}=6x+3\,.\,\delta x,$$

$$\therefore\ \frac{d}{dx}(3x^2)=\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}=\underset{\delta x\to 0}{\mathrm{Lt}}(6x+3\,.\,\delta x)=6x.$$

Special results:

(i) If $y = C$ (a constant), then $\dfrac{dy}{dx} = \dfrac{d}{dx}(C) = 0.$

Since y is constant, its value never alters whatever changes are made in the value of x.

$$\therefore\ \delta y = 0 \text{ for all values of } \delta x,$$

$$\therefore\ \frac{\delta y}{\delta x} \text{ is always zero: } \therefore\ \frac{dy}{dx} = 0.$$

Note that the graph of $y = C$ is a straight line parallel to OX and $\therefore$ its gradient is zero.

(ii) If $y = ax$, where a is constant, $\dfrac{dy}{dx} = \dfrac{d}{dx}(ax) = a$;

we have $y = ax$ and $y + \delta y = a(x + \delta x)$.

$$\therefore\ \delta y = a\,\delta x,$$

$$\therefore\ \frac{\delta y}{\delta x} = a,$$

$$\therefore\ \frac{dy}{dx} = a.$$

(iii) If $y = x^3$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^3) = 3x^2$;

we have $y = x^3$ and $y + \delta y = (x + \delta x)^3$

$$= x^3 + 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3,$$

$$\therefore\ \delta y = 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3,$$

$$\therefore\ \frac{\delta y}{\delta x} = 3x^2 + 3x\,\delta x + (\delta x)^2,$$

$$\therefore\ \frac{dy}{dx} = \operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2.$$

(iv) If $y = \dfrac{1}{x}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$;

we have $y = \dfrac{1}{x}$ and $y + \delta y = \dfrac{1}{x + \delta x}$,

$$\therefore \delta y = \frac{1}{x+\delta x} - \frac{1}{x} = \frac{x-(x+\delta x)}{x(x+\delta x)} = \frac{-\delta x}{x^2 + x \, . \, \delta x},$$

$$\therefore \frac{\delta y}{\delta x} = -\frac{1}{x^2 + x \, . \, \delta x},$$

$$\therefore \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = -\frac{1}{x^2}.$$

We have now proved that

$$\frac{d}{dx}(C) = 0, \quad \frac{d}{dx}(x) = 1, \quad \frac{d}{dx}(x^2) = 2x, \quad \frac{d}{dx}(x^3) = 3x^2.$$

These results are all included in the following general result.

If $y = x^n, \ \dfrac{dy}{dx} = \dfrac{d}{dx}(x^n) = nx^{n-1}$

This is true for all values of n, but, to start with, we shall prove it only for the case where n is a positive integer.

The Binomial Theorem states that, if n is a positive integer,

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{1 \, . \, 2} x^{n-2}h^2 + \ldots + h^n.$$

If $y = x^n$, $\delta y = (x+\delta x)^n - x^n$

$$= x^n + nx^{n-1} \, . \, \delta x + \frac{n(n-1)}{1 \, . \, 2} x^{n-2} \, . \, (\delta x)^2 + \ldots + (\delta x)^n - x^n,$$

$$\therefore \frac{\delta y}{\delta x} = nx^{n-1} + \delta x \, [F],$$

where F remains finite when $\delta x \to 0$,

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = nx^{n-1}.$$

It should be noted that the result $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$ obtained above is included in this formula: for

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)\, x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

The statement $\dfrac{d}{dx}(x^n) = nx^{n-1}$ is equivalent to saying that

$$\underset{\delta x \to 0}{\mathrm{Lt}} \frac{(x+\delta x)^n - x^n}{\delta x} = nx^{n-1}.$$

It follows from this that, if C is any constant, $\dfrac{d}{dx}(Cx^n) = C \,.\, nx^{n-1}$.

For $$\frac{d}{dx}(Cx^n) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{C(x+\delta x)^n - Cx^n}{\delta x} = C \,.\, \underset{\delta x \to 0}{\mathrm{Lt}} \frac{(x+\delta x)^n - x^n}{\delta x}$$
$$= C \,.\, nx^{n-1}.$$

If the function of x which is to be differentiated consists of more than one term, we can differentiate it from first principles as follows.

Suppose $$y = ax^2 + bx + c.$$

Let $P(x, y)$ be a point on the curve and $Q(x+\delta x, y+\delta y)$ another point near to P.

Then $$y + \delta y = a(x+\delta x)^2 + b(x+\delta x) + c$$
$$= ax^2 + 2ax \,.\, \delta x + a(\delta x)^2 + bx + b(\delta x) + c.$$
$$\therefore \; \delta y = 2ax \,.\, \delta x + b \,.\, \delta x + a(\delta x)^2,$$
$$\therefore \; \frac{\delta y}{\delta x} = 2ax + b + a \,.\, \delta x,$$
$$\therefore \; \frac{d}{dx}(ax^2 + bx + c) = \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = 2ax + b.$$

It should be noted that the result obtained is the same as if each term had been differentiated separately: thus

$$\frac{d}{dx}(ax^2 + bx + c) = \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx) + \frac{d}{dx}(c).$$

This is true for all sums and differences. Thus

$$\frac{d}{dx}(7x^3 - 5x^2 + 10x) = \frac{d}{dx}(7x^3) - \frac{d}{dx}(5x^2) + \frac{d}{dx}(10x)$$
$$= 7\frac{d}{dx}(x^3) - 5\frac{d}{dx}(x^2) + 10\frac{d}{dx}(x)$$
$$= 7 \times 3x^2 - 5 \times 2x + 10 \times 1$$
$$= 21x^2 - 10x + 10.$$

Successive Differentiation

If $y=x^3$, $\qquad \dfrac{dy}{dx}=\dfrac{d}{dx}(x^3)=3x^2,$

and $\qquad \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dx}(3x^2)=6x.$

The expression $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is written $\dfrac{d^2y}{dx^2}$ and is called the *second differential coefficient* of y with respect to x.

Similarly $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ is written $\dfrac{d^3y}{dx^3}$ and is called the *third differential coefficient*, and so on.

Reverse Process

Example. Find y when $\dfrac{dy}{dx}=x^2$.

Remembering that when $y=x^3$ we have $\dfrac{dy}{dx}=3x^2$, it is clear that $y=\dfrac{x^3}{3}$ will give $\dfrac{dy}{dx}=x^2$, but it must be remembered that $y=\dfrac{x^3}{3}+c$ (where c is a constant) would also satisfy the equation, and the result should be given in this form.

EXAMPLES IIIa

1. Find expressions for δy, $\dfrac{\delta y}{\delta x}$, $\dfrac{dy}{dx}$ in the following cases:

(i) $y=4x+5$. (ii) $y=5x^2$. (iii) $y=x^2-3x+2$.

(iv) $y=\dfrac{5}{x}$. (v) $y=(x+3)^2$.

2. Find the numerical values of the gradients to the following curves at the points named:

(i) $y=2x^2$ at $(2, 8)$.

(ii) $y=x^2-2x$ at $(3, 3)$, $(1, -1)$, $(0, 0)$.

(iii) $y=\dfrac{12}{x}$ at $(3, 4)$.

3. Fig. 38 represents the graph of $y=\frac{1}{2}x^2$; calculate $\angle PTN$ if PT is a tangent and if P is (i) $(2, 2)$, (ii) $(1, \frac{1}{2})$.

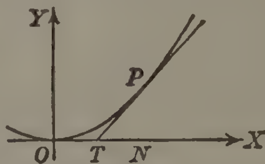

Fig. 38.

Differentiate with respect to x the expressions in Examples 4—27:

4. x^4. **5.** $3x^7$. **6.** $2x^3-3$. **7.** $5x^2+4x$. **8.** $2x^8-8x+5$.

9. $(3x)^2$. **10.** $1-x$. **11.** $2+x-x^3$. **12.** x^p. **13.** x^{2q}.

14. $\dfrac{3x^2}{4}$. **15.** $0{\cdot}1x-0{\cdot}4x^4$. **16.** $(2x-1)^2$. **17.** ax^3. **18.** bx.

19. $(2-x)(x-1)$. **20.** πx^2. **21.** $\frac{4}{3}\pi x^3$. **22.** $(3x^2)(5x^3)$.

23. $\frac{3}{4}x^4-\frac{1}{2}x^2+\frac{1}{3}$. **24.** $(3+x)(5+x^2)$. **25.** $2x+\dfrac{3}{x}$.

26. $\dfrac{x^3-1}{x}$. **27.** $\dfrac{1}{2x}-3-x^3$.

28. Find $\dfrac{ds}{dt}$ given that (i) $s=5+4t^2$, (ii) $s=3t+16t^2$, (iii) $s=ut+\frac{1}{2}gt^2$.

29. If $P=3s^2-3s$, find $\dfrac{dP}{ds}$ when $s=1$.

30. If $y=x^3-3x^2$, find $\dfrac{dy}{dx}$ when $x=2$.

31. Find the points on $y=x^3-3x$ at which the gradient is zero.

32. Find the points on $y=x^3-2x^2+x$ at which the tangent is parallel to Ox.

33. Find the gradients of $y=x^3-6x^2+11x-6$ at the points $(1,0)$, $(2,0)$, $(3,0)$.

34. Find y for the following values of $\dfrac{dy}{dx}$:

(i) x.	(ii) 1.	(iii) $3x$.	(iv) x^3.
(v) $2x^3$.	(vi) $5x^4$.	(vii) $1-x$.	(viii) $3x^2+2$.
(ix) $2\pi x$.	(x) $4\pi x^2$.	(xi) $-\dfrac{1}{x^2}$.	(xii) $\dfrac{2}{x^2}$.

35. If $\dfrac{dv}{dt}=6$, find v given that, when $t=0$, $v=4$.

36. If $\dfrac{ds}{dt}=u+at$, find s given that u and a are constants and that $s=0$ when $t=0$.

37. If $y=x^4$, prove that $x\dfrac{dy}{dx}=4y$.

38. If $y=x^2-x$, prove that $\dfrac{dy}{dx}=1+\dfrac{2y}{x}$.

39. If $y=x^4$, prove that $4y\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=3\left(\dfrac{dy}{dx}\right)^2$.

40. By drawing a figure, interpret geometrically the relation

$$\frac{d}{dx}(3x^2) \equiv \frac{d}{dx}(3x^2+2).$$

41. Illustrate by a figure the identity $\frac{d}{dx}(4x^3) \equiv 4\frac{d}{dx}(x^3)$.

42. Draw freehand a portion of the curve passing through the origin whose equation is $y=f(x)$ given

(i) $\frac{dy}{dx}=1, 0, -1$ for $x=0, 1, 2$;

(ii) $\frac{dy}{dx}=0$ at each of the points $(0, 0)$, $(2, -4)$, $(4, 0)$ and is -1 when $x=1$.

43. (i) Express $\frac{x^2+4x-5}{x-1}$ in terms of h if $x=1+h$.

(ii) Evaluate $\underset{x\to 1}{\text{Lt}} \frac{x^2+4x-5}{x-1}$.

44. (i) Prove that, if $x \neq a$, $\frac{x^{10}-a^{10}}{x-a}=x^9+x^8a+x^7a^2+\ldots+xa^8+a^9$.

(ii) Evaluate $\underset{\delta x\to 0}{\text{Lt}} \frac{(x+\delta x)^{10}-x^{10}}{(x+\delta x)-x}$.

(iii) Use (ii) to find $\frac{d}{dx}(x^{10})$.

45. (i) What is the quotient when x^n-a^n is divided by $x-a$, where n is a positive integer?

(ii) Evaluate $\underset{x\to a}{\text{Lt}} \frac{x^n-a^n}{x-a}$.

(iii) Evaluate $\underset{\delta x\to 0}{\text{Lt}} \frac{(x+\delta x)^n-x^n}{\delta x}$ and $\frac{d}{dx}(x^n)$.

46. At what points on the curve $y=x^3-2x$ is the gradient 1?

47. Find $\frac{d^2y}{dx^2}$ if

(i) $y=x^4+3$, (ii) $y=2x^3-7x^2+5x$, (iii) $y=(2x-1)(3x+1)$.

48. Find $\frac{d^2s}{dt^2}$ if

(i) $s=10t-16t^2$, (ii) $s=ut+\frac{1}{2}ft^2$ where u, f are constants.

49. Find $\frac{d^2y}{dx^2}$ if $y=mx+c$ where m, c are constants.

50. If $y=px^2+qx$ where p, q are constants, prove that

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$$

51. The radius of the circle which has closest contact with the curve $y=f(x)$ at the point (x, y) is $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \div \frac{d^2y}{dx^2}$. Calculate the radius of this circle for the curve $y=x^2$ (i) at the origin, (ii) at the point $(\frac{2}{3}, \frac{4}{9})$.

52. What is the value of $\frac{d^2y}{dx^2}$ when $\frac{dy}{dx}=0$ for the curve

$$\text{(i) } y=x^2-x, \quad \text{(ii) } y=x^3-6x^2?$$

53. Verify the identity $\frac{d}{dx}\left(x\frac{dy}{dx}\right) \equiv x\frac{d^2y}{dx^2}+\frac{dy}{dx}$ when $y=x^3-3x$.

Equation of a straight line

If we are given the gradient m of a straight line and the coordinates of a fixed point on it, e.g. A (3, 4), we are required to find the relation between the coordinates x and y of any point P on the line.

Draw AR parallel to OX; then from the figure

$$\frac{RP}{AR}=m \text{ but } RP=y-4,$$

and $\quad AR=x-3, \quad \therefore \frac{y-4}{x-3}=m,$

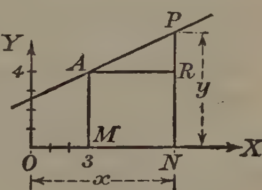

Fig. 39.

or $y-4=m(x-3)$ is the required relation.

If the coordinates of A are x_1, y_1, the relation would be

$$y-y_1=m(x-x_1).$$

Note. x_1y_1, x_2y_2, etc. are used for fixed points, and are *constants.*

Gradient of perpendicular lines

If the slope of a line is θ, the angle a perpendicular to this line makes with the positive direction of OX will be $(90+\theta)$.

If the gradient of TP is m and the gradient of GP is m', then $\tan\theta=m$ and $\tan(90+\theta)=m'$. But

$$\tan(90+\theta)=-\cot\theta=-\frac{1}{\tan\theta}.$$

$$\therefore m'=-\frac{1}{m}.$$

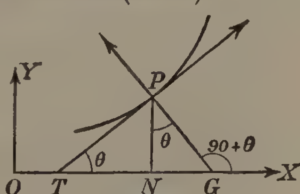

Fig. 40.

Tangents and Normals

A line at right angles to a tangent through its point of contact P is called a Normal to the curve. If NP is the ordinate of P, TN is called the subtangent and NG the subnormal. See Fig. 40.

From the figure it is clear that the subtangent TN

$$= NP \cot\theta = \frac{y}{\dfrac{dy}{dx}},$$

also the subnormal $= NP \tan NPG = y \tan\theta = y \cdot \dfrac{dy}{dx}$.

Example.

Find the equation of the tangent and the normal at $(1, 3)$ on the curve $y=3x^2$.

The gradient of the tangent is $\dfrac{dy}{dx}=6x$,

$$\therefore \text{ at the point } (1, 3) \text{ it is } 6,$$

$$\therefore \text{ the equation of the tangent is } y-3=6(x-1),$$

and since the gradient of the normal at that point is $-\frac{1}{6}$, the equation of the normal is

$$y-3=-\tfrac{1}{6}(x-1).$$

EXAMPLES III b

1. Find the equation of the tangent to $y=3x^3+2x+1$ at the point where $x=1$.

2. The tangent to $y=1-3x^3$ at the point (a, b) is equally inclined to the axes.

Find the positive values of a and b and hence the equation of the tangent at that point.

3. Find the equations of the tangents to $y=(x+2)(x-1)$ at the points where the curve cuts OX.

4. At what point on the curve $y=2x^2-x+1$ is the tangent parallel to the line $y=3x+4$?

Find the equation of this tangent.

5. Find the equations of the tangent and the normal to the curve $y=2x^3+5$ at the point on the curve whose abscissa is $\frac{1}{2}$.

6. Find the equations of the tangents to $2y=4x^3-x^2-4x+14$ whose gradients are zero.

7. Find the lengths of the subtangent and subnormal at the point $(1, 4)$ on the curve $y=3x+x^2$.

8. Find the equation of the tangent to $y=5x^2$ which passes through the point $(2, 0)$.

9. Find the length of the subnormal at the point where $y=5x-4x^3+2$ cuts the y-axis.

10. Find the lengths of the tangent and normal at the point $(1, \frac{1}{2})$ on $4y=x+x^2$.

CHAPTER IV

MAXIMA AND MINIMA

IN this chapter and in fact throughout this book we shall assume that the functions we are dealing with are continuous, at any rate for the range of values considered: that is to say, their graphs have no breaks or gaps: we shall also assume there are no abrupt changes of direction in the curves which represent the functions dealt with.

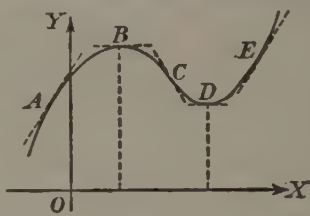

Fig. 41.

If we place a ruler in the position of the tangent at successive points on the graph given in Fig. 41, we note that at A the gradient is positive (this means that, from A to B, y is increasing as x increases and $\frac{dy}{dx}$ is positive). From A to B the gradient remains positive but decreases in value until at B it becomes zero. After B, the gradient is negative, attains its greatest negative value near C, then becomes less negative until at D it is again zero (this means that, from B to D, y is decreasing as x increases and $\frac{dy}{dx}$ is negative, its greatest negative value being near C). From D onwards, the gradient is positive and steadily increases.

These results may be tabulated as follows:

	from A to B	at B	from B to D	at D	from D to E
the gradient $= \frac{dy}{dx}$ is	+	0	−	0	+

At B the ordinate is greater than the neighbouring ordinates on either side, and at D the ordinate is less than the neighbouring ordinates on either side.

B and D are called *turning points*: at B, the function is said to have a *maximum* value and at D a *minimum* value.

We see then that both at a maximum and at a minimum $\frac{dy}{dx}=0$. But in passing through a maximum $\frac{dy}{dx}$ changes from $+$ to $-$, while in passing through a minimum $\frac{dy}{dx}$ changes from $-$ to $+$. [See Fig. 42.]

Suppose the equation of the curve in Fig. 41 is

$$y = 2x^3 - 9x^2 + 12x + 4.$$

Then the gradient

$$= \frac{dy}{dx} = 6x^2 - 18x + 12$$

$$= 6(x^2 - 3x + 2),$$

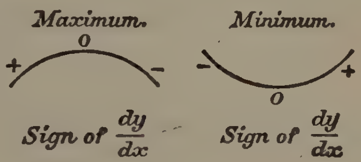

Fig. 42.

$\therefore$ the gradient is zero if $x^2 - 3x + 2 = 0$,

$$\text{or } (x-1)(x-2) = 0,$$

i.e. when $x = 1$ or $x = 2$.

If x is slightly less than 1, $\frac{dy}{dx} = 6(x-1)(x-2)$ is positive.

If x is slightly more than 1, $\frac{dy}{dx} = 6(x-1)(x-2)$ is negative.

$\therefore x = 1$ gives a maximum value of y,

$\therefore y = 2 - 9 + 12 + 4 = 9$ is a maximum value.

Similarly, we find that just before $x = 2$, $\frac{dy}{dx}$ is negative and just after $x = 2$, $\frac{dy}{dx}$ is positive.

$\therefore x = 2$ gives a minimum value of y,

$\therefore y = 2 \times 8 - 9 \times 4 + 12 \times 2 + 4 = 16 - 36 + 24 + 4 = 8$ is a minimum value.

It is often, however, easier to distinguish between maximum and minimum positions in the following way:

Fig. 42 shows that in passing through a maximum, $\frac{dy}{dx}$ changes from + to – through zero and is therefore decreasing. Consequently the gradient of $\frac{dy}{dx}$, i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$, is negative. But in passing through a minimum, $\frac{dy}{dx}$ changes from – to + through zero and is therefore increasing, and so the gradient of $\frac{dy}{dx}$, i.e. $\frac{d^2y}{dx^2}$, is positive.

Hence at a maximum, $\frac{d^2y}{dx^2}$ is negative,

at a minimum, $\frac{d^2y}{dx^2}$ is positive.

In the above example we found that

$$\frac{dy}{dx} = 6x^2 - 18x + 12,$$

$$\therefore \frac{d^2y}{dx^2} = 12x - 18,$$

$$\therefore \text{ for } x = 1, \frac{d^2y}{dx^2} = -6 \text{ and } \therefore \text{ is negative.}$$

But $$\text{for } x = 2, \frac{d^2y}{dx^2} = +6 \text{ and } \therefore \text{ is positive,}$$

showing that $x = 1$ gives a maximum value of y and $x = 2$ gives a minimum value of y.

Derived Curves

Suppose in Fig. 43 the curve A is the graph of the function

$$y = f(x) \equiv ax^3 + bx^2 + cx + d.$$

The signs of the gradients of this curve, as drawn, are by inspection:

Values of x	0	OP	OQ	OR	OS
Sign of gradient of $f(x)$	+	0	–	0	+

Now the gradient of $f(x)$ is $\frac{d}{dx}[f(x)] = 3ax^2 + 2bx + c$ and is usually written $f'(x)$.

Curve B is a rough graph of $y = f'(x)$, drawn from the table set out above, and is called the *First Derived Curve.*

Similarly, we have by inspection:

Values of x	0	$O'P$	$O'Q$	$O'R$	$O'S$
Sign of gradient of $f'(x)$	$-$	$-$	0	$+$	$+$

Now the gradient of $f'(x)$ is

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}[3ax^2 + 2bx + c] = 6ax + 2b,$$

and is usually written $f''(x)$.

Curve C (in this case a straight line) is a rough graph of $y = f''(x)$, drawn from the above table, and is called the *Second Derived Curve.* Similarly we obtain the third derived curve

$$y = f'''(x)$$

$$\equiv \frac{d}{dx}(6ax + 2b) = 6a,$$

which is a straight line parallel to the x-axis.

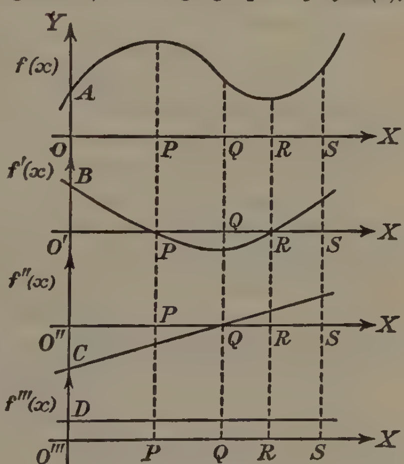

Fig. 43.

Fig. 43, which represents the graph of a function and the derived curves, illustrates the following points:

(i) Where the curve A has a maximum or minimum, the gradient of $f(x)$ is zero, and consequently for these values of x the curve B cuts the x-axis.

(ii) When A passes through a *maximum*, B changes from + to −, i.e. it decreases, so its gradient is negative and therefore C lies below the x-axis.

(iii) When A passes through a *minimum*, B changes from − to +, i.e. it increases, so its gradient is positive and therefore C lies above the x-axis.

(iv) For the value of x at Q, the curve A changes from being concave downwards to being concave upwards, i.e. it has been bending one way and at Q starts to bend the other way; its gradient has reached its greatest negative value, so that B reaches a minimum at Q. Since the gradient of B is zero at Q, the curve C crosses its x-axis at Q. The graph of $f(x)$ is said to have a *point of inflexion* for the value of x at Q.

There are three other points which should be noted:

(1) Fig. 44 represents the graph of a function which attains maximum values at C, E and attains minimum values at B, D, F. But yet the value at the maximum E is less than the value at the minimum B: and all these values are less than the value of the function at A which is not a maximum value of the function in the sense in which these terms have been used. Positions are called maximum or minimum positions if the value of the function at such a point is respectively greater or less than the values in neighbouring positions on *both* sides.

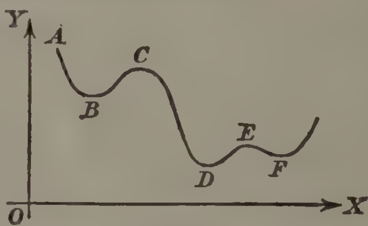

Fig. 44.

The word "stationary" is used for any point at which $\frac{dy}{dx}=0$.

(2) A value of x for which $\frac{dy}{dx}=0$ does not necessarily give either a maximum or a minimum. In Fig. 45, at the stationary

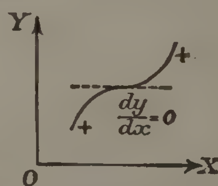

Fig. 45.

point, $\frac{dy}{dx}=0$, but just before it and just after it $\frac{dy}{dx}$ is positive. It is therefore essential that $\frac{dy}{dx}$ should also change sign.

(3) If $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}$ is positive, y is a minimum; but y *may* be a minimum (or a maximum or neither) when $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both zero; e.g., at (0, 0) on $y=x^4$ we have $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=0$, and for this function y is a minimum at (0, 0).

Example 1.

A farmer has sufficient hurdles to make a rectangular pen of perimeter 200 feet. What dimensions will give an enclosure of maximum area.

Let the required length be x feet, then the breadth is $100-x$ feet and the area is A sq. feet, where

$$A=x(100-x)=100x-x^2.$$

If we were to draw a graph of A and x, the value of A, given by NP, is greatest when the gradient at P is zero.

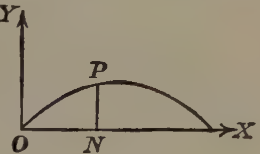

Fig. 46.

But $$\frac{dA}{dx}=100-2x,$$

$\therefore \frac{dA}{dx}$ is zero when $x=50$.

To show that $x=50$ gives a maximum and *not* a minimum value of A we can say

either when $x<50$, $\frac{dA}{dx}$ is positive,

when $x>50$, $\frac{dA}{dx}$ is negative.

$\therefore \frac{dA}{dx}$ changes from + to − in moving through $x=50$,

$\therefore x=50$ gives a maximum value of A.

or $\frac{d^2A}{dx^2}=-2$ and is therefore negative,

$\therefore x=50$ gives a maximum value of A.

$\therefore$ the maximum enclosure is a square of side 50 feet.

Example 2.

Find the cylinder of greatest volume that can be inscribed in a circular cone of height h inches and base-radius a inches.

If the radius of the cylinder is $DE=x$ inches, the height of the cylinder is EF.

Now $$BE=BD-ED=a-x,$$

and $$\frac{FE}{EB}=\frac{AD}{DB}, \therefore \frac{FE}{a-x}=\frac{h}{a},$$

$$\therefore FE=\frac{h}{a}(a-x).$$

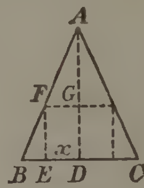

Fig. 47.

$\therefore$ volume, V cu. ins., of cylinder is given by

$$V=\pi x^2 \cdot \frac{h}{a}(a-x)=\frac{\pi h}{a}(ax^2-x^3).$$

Put $y=ax^2-x^3$; then V is greatest when y is greatest since $\frac{\pi h}{a}$ is a constant.

$$\frac{dy}{dx}=2ax-3x^2=x(2a-3x),$$

$$\therefore \frac{dy}{dx}=0 \text{ if } x=0 \text{ or } x=\frac{2a}{3}.$$

But $$\frac{d^2y}{dx^2}=2a-6x,$$

$$\therefore \frac{d^2y}{dx^2}=2a \text{ if } x=0 \text{ and } \frac{d^2y}{dx^2}=-2a \text{ if } x=\frac{2a}{3},$$

$$\therefore x=\frac{2a}{3} \text{ gives a } \textit{maximum} \text{ value for } y.$$

$\therefore$ the maximum value of V is

$$\pi\left(\frac{2a}{3}\right)^2 \cdot \frac{h}{a}\left(a-\frac{2a}{3}\right)=\frac{4}{27}\pi a^2 h \text{ cu. ins.}$$

Note that it is often possible to distinguish between a maximum or minimum by geometrical considerations. In the above example, the volume of the cylinder is obviously zero both when $x=a$ and when $x=0$ and is positive for intermediate values; consequently there must be a *maximum* volume for some value of x between $x=a$ and $x=0$.

Example 3.

One side of a triangular enclosure is formed by part of a hedge AB A fence is constructed perpendicular to AB as far as some point P and then from P to some point Q in AB. The total length of the fence is 90 feet; what length of AP will make the enclosure as large as possible?

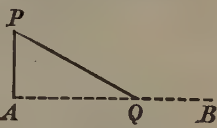

Fig. 48.

Let AP equal x feet, $\therefore PQ=90-x$ feet.

$\therefore AQ=\sqrt{[(90-x)^2-x^2]}=\sqrt{[8100-180x]}$ feet.

$\therefore$ area of $\Delta\ PAQ=\frac{1}{2}x\,.\sqrt{[8100-180x]}$ sq. feet.

This is a maximum if $y=x^2(8100-180x)$ is a maximum.

Now $$y=180\,(45x^2-x^3),$$

$$\therefore \frac{dy}{dx}=180\,(90x-3x^2)=180x\,(90-3x),$$

$$\therefore \frac{dy}{dx}=0 \text{ if } x=0 \text{ or } x=30.$$

But $\dfrac{d^2y}{dx^2}=180\,(90-6x)$ and is $\therefore$ negative if $x=30$.

$\therefore$ the enclosure is largest when $AP=30$ feet.

It should be noted that the rules so far given do not enable us to differentiate $x\sqrt{[8100-180x]}$: but this difficulty has been avoided by finding the value of x for which the square of this expression is a maximum.

EXAMPLES IV

1. Sketch the graph of $y=4x-x^2$ for positive values of x. What is the sign of $\dfrac{dy}{dx}$ when $x=1, 3, 5$? Does $\dfrac{dy}{dx}=0$ give a maximum or minimum value of y? What is the value of $\dfrac{d^2y}{dx^2}$? Sketch the first derived curve.

2. Sketch the graph of $y=1+x^3$. What is the sign of $\dfrac{dy}{dx}$ when $x=-1, 1$? Does $\dfrac{dy}{dx}=0$ give a turning value of y? What is the value of $\dfrac{d^2y}{dx^2}$ when $\dfrac{dy}{dx}=0$? Sketch the first and second derived curves.

3. Make a table showing the signs of $\frac{dy}{dx}$ for different portions of the graphs given in Figs. 49, 50, 51.

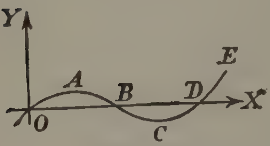

Fig. 49.

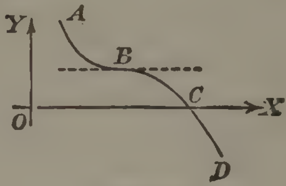

Fig. 50.

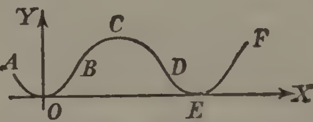

Fig. 51.

4. Sketch roughly the first and second derived curves of the graphs given in Figs. 49, 50, 51.

5. Sketch roughly the graph of $y=f(x)$ given that

(i) if $x<1$, $\frac{dy}{dx}$ is positive; at $x=1$, $\frac{dy}{dx}=0$; if $x>1$, $\frac{dy}{dx}$ is negative;

(ii) if $x<1$, $\frac{dy}{dx}$ is negative; at $x=1$, $\frac{dy}{dx}=0$; if $x>1$, $\frac{dy}{dx}$ is positive;

(iii) if $x=1$, $\frac{dy}{dx}=0$; for all other values of x, $\frac{dy}{dx}$ is negative.

6. Sketch the graph of $y=x(x-3)(x-8)$ between $x=0$ and $x=10$. What is the sign of $\frac{dy}{dx}$ when $x=1, 2, 5, 7$? For what values of x is y stationary? Distinguish between them. For what value of x does $\frac{d^2y}{dx^2}$ vanish? Sketch the first and second derived curves.

7. Find the turning-points of each of the following functions and state whether the function is a maximum or minimum:

(i) $2x-x^2$; (ii) $11-6x+x^2$; (iii) x^3-3x;
(iv) $4x^4-2x^2$; (v) x^4-4x^3; (vi) $(x^2-10)(x^2+2)$;
(vii) $(x^2+10)(x^2+2)$; (viii) x^3-3x^2+3x; (ix) $x^2+\frac{16}{x}$.

8. A particle is projected vertically upwards and rises s feet in t seconds where $s=80t-16t^2$. What is the greatest height it attains?

9. One side of a rectangular enclosure is formed by a hedge, the total length of fence forming the other three sides is 300 yards. What is the maximum area of the enclosure?

10. The ends of a beam AB, 24 feet long, are built in at the same level to two walls; AB carries a load which is heavy compared with the weight of the beam at a point 6 feet from A. The sag at a point x feet from A is y inches where $y=\frac{1}{160}x^2(12-x)$. Find the maximum sag.

11. A particle is projected horizontally into a resisting medium and travels s feet in t seconds where $s=15t-\frac{1}{5}t^3$. How far does it penetrate?

12. The parcel post regulations require that the sum of the length and girth of a parcel shall not exceed 6 feet. What is the maximum volume of a parcel with a square base?

13. Fig. 52 represents the wall of a workshop pierced with six equal windows spaced so that all the dotted lines are 5 feet long. The total area of window space is to be 288 sq. feet. What should be the dimensions of each window to give the minimum amount of brickwork?

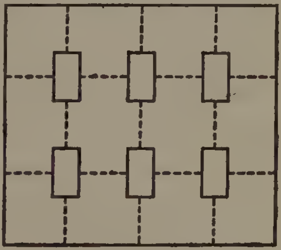

Fig. 52.

14. Fig. 53 represents the graphs of $y=\frac{1}{3}x^2$ and $y=2x$; PQ is perpendicular to Ox; what is its maximum length?

15. Fig. 54 represents a skeleton box on a square base $ABCD$ made of 17 portions of wire; if the total length of wire is 18 feet, what is the maximum volume of the box?

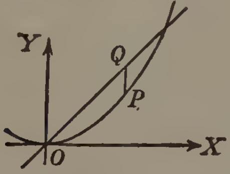

Fig. 53

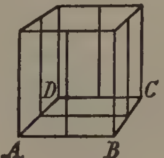

Fig. 54.

16. If $x-3y=2$, find the least value of x^2+y^2.

17. $ABCD$ is a rectangle. $AB=9''$, $BC=4''$. If a point P moves along DC, find its distance from D when AP^2+PB^2 is a minimum.

18. Fig. 55 represents the graph of $y=x^2(3-x)$. A line is drawn parallel to Ox to touch the curve at P and cut it again at Q. Find the length of PQ. Find also the value of x for which $\frac{d^2y}{dx^2}$ changes sign, and explain what this means.

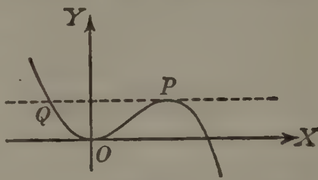

Fig. 55.

19. A rectangular sheet of metal is 8 by 3 feet: equal squares are cut out at each of the corners and the flaps are then folded up to form an open rectangular box. Find its maximum volume.

20. If a compound circuit of n cells is arranged in rows with p cells in each row, and if each cell is of resistance r ohms and electromotive force E volts, and if the external resistance is R ohms, the current is I amperes where

$$I=\frac{E}{\frac{pr}{n}+\frac{R}{p}}.$$

Regarding E, R, r, n as constants, find (i) the value of p for which $\frac{pr}{n}+\frac{R}{p}$ is least, (ii) the greatest value of I.

21. If a circular cylinder is cut from a sphere, prove that the volume is greatest when the height equals $\sqrt{2}\times$ the radius of the base.

22. Fig. 56 represents a curve called the Witch of Agnesi whose equation is

$$y=3\sqrt{\left\{\frac{3-x}{x}\right\}}.$$

Find the maximum area of the triangle ONP.

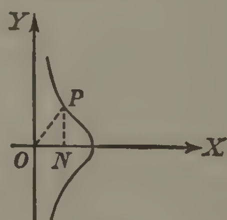

Fig. 56.

23. An ordinary match box and cover consists of an open box twice as wide as it is high which slides inside a cover. What should be the ratio of the length to the height, if for a given area of matchwood the volume is a maximum?

24. A skeleton cylinder is formed of two equal circular pieces of wire joined by four separate equal straight pieces. For a given length of wire, show that the volume is greatest when twice as much is used for the circular as for the straight pieces.

25. A given current I flows into a divided circuit, the portions of which are of resistances R_1, R_2, and the currents I_1, I_2. The heat generated H is given by the equation $H = I_1^2 R_1 + I_2^2 R_2$, also $I = I_1 + I_2$. The current is distributed so as to make the heat generated a minimum. Treating R_1, R_2, I as constants find $\frac{I_1}{I_2}$ and H in terms of R_1, R_2, I.

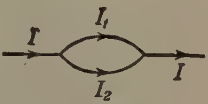

Fig. 57.

26. $ABCD$ is a sheet of cardboard, 2 feet square; creases PQ, RS are made parallel to AD and equidistant from the centre of the square and the flaps are folded up to form a triangular prism with open ends. Where must the creases be made to give a maximum volume?

27. AB and CD are two cables of equal length. The strain that any section of AB can stand varies as the cube of its distance from B, and the strain that any section of CD can stand varies as the cube of its distance from D: further, the strain that can be carried at A is 4 times that at C. The cables are woven together so that D coincides with A and C with B. Where is the weakest point in the composite cable?

28. Fig. 58 represents the path of a shell fired with velocity u feet per sec. in a direction making an angle a with the horizontal. The equation of the curve is

$$y = mx - \frac{16x^2}{u^2}(1 + m^2),$$

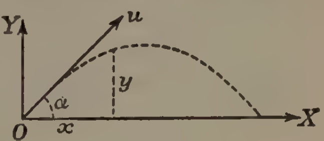

Fig. 58.

where $m = \tan a$. What problem would be solved by

(i) putting $\frac{dy}{dx} = 0$ treating m and u as constant,

(ii) putting $\frac{dy}{dm} = 0$ treating x and u as constant?

If $u = 800$, can an elevation be found such that the shell will clear a ridge 3500 feet high at a distance of 16,000 feet?

29. ABC is a triangle of area 10 sq. ins.; P is a point on AB such that $\frac{AP}{AB} = x$ where $x > \frac{1}{2}$; PQ, PR are drawn parallel to AC, BC cutting CB, AC at Q, R respectively; RS is drawn parallel to AB cutting BC at S; prove that the area of $\Delta\ APR = 10x^2$ sq. ins. and the area of

$\triangle PBQ = 10(1-x)^2$ sq. ins. And find the position of P for which the area of $PQRS$ is a maximum. (C.S.C.)

30. The efficiency of a certain screw is $\dfrac{12t-5t^2}{5+12t}$ where t is the tangent of its pitch.

Find its maximum efficiency. [Put $5+12t=x$.]

31. A uniform sheet of metal, weight 180 lbs., is in the form of an isosceles triangle ABC with its base BC suspended horizontally and a weight of 20 lbs. attached to A: the altitude $AD=12$ feet. PQ is a line parallel to BC cutting AB, AC at P, Q and at distance x feet from A. Find in terms of x the weight of the part below PQ and the strain per foot length of PQ, given $BC=6$ feet. For what position of PQ is this least?

32. A cross-channel ferry is constructed so as to transport a fixed number of tons across each way per day. If the cost of construction of the ferry without the engines varies as the load, and the cost of the engines varies as the load and the cube of the speed, prove that the total cost of construction is least when twice as much money is spent on the ferry as on the engines. [Neglect the time of loading and unloading and assume the ferry runs continuously.]

CHAPTER V

THE DERIVATIVE AS A RATE-MEASURER

If a stone is dropped and falls 64 feet in 2 seconds, it is not true to say that its velocity is 32 feet per sec. We could call 32 feet per sec. its average speed, but as it is getting faster all the time, this average speed would be a misleading description of (say) its final speed. Again when we say that a train passing us is travelling 60 miles an hour we do not mean that it has actually travelled 60 miles in the previous hour nor that it will travel 60 miles in the next hour. What we mean is that it will travel *about* 1 mile in the next minute, or with less percentage error *about* $\frac{1}{2}$ mile in the next 30 seconds, or with still less percentage error about 88 feet in the next second. In order to discover its speed, we should measure the distance it went in as short a time as our instruments would allow and work out an average velocity for this interval: the shorter the interval, the more accurately we should regard this average velocity as representing the train's speed at the moment it passed us.

Example 1.

A body moves so that it travels s feet in t seconds where $s=3t^2$. Find its velocity at any moment.

Suppose it travels $s+\delta s$ feet in $t+\delta t$ seconds.

Then $$s+\delta s=3(t+\delta t)^2=3t^2+6t\,.\,\delta t+3(\delta t)^2.$$

But $$s=3t^2;$$

$$\therefore\ \delta s=6t\,.\,\delta t+3(\delta t)^2.$$

$\therefore$ it travels an extra δs feet in an extra δt seconds where

$$\delta s=6t\,.\,\delta t+3(\delta t)^2;$$

$\therefore$ its average speed over this interval is

$$\frac{\delta s}{\delta t}=6t+3\,.\,\delta t \text{ feet per sec.}$$

The smaller δt becomes, the closer does the average speed approximate to our meaning of the phrase "velocity after t seconds": and we define the speed after t seconds to be the limit of $\frac{\delta s}{\delta t}$ when $\delta t \to 0$,

$$\therefore \text{ velocity after } t \text{ seconds} = \frac{ds}{dt} = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta s}{\delta t} = 6t \text{ feet per sec.}$$

Thus, for example, after 5 seconds its speed is 30 feet per sec.

In Dynamics it is usual to use the notation $\dot{x}$ for $\frac{dx}{dt}$, $\dot{\theta}$ for $\frac{d\theta}{dt}$, $\ddot{x}$ for $\frac{d^2x}{dt^2}$ and $\ddot{\theta}$ for $\frac{d^2\theta}{dt^2}$.

The velocity of a body means the rate at which the distance it travels increases with respect to the time. We see then that the differential coefficient $\frac{ds}{dt}$ measures the rate at which s increases with respect to t.

It is important, however, to realise that we can speak of the rate of increase of one variable with respect to another without any reference to time at all.

Suppose for example a spherical snowball of radius r inches weighs W ozs. where $W = 0{\cdot}5r^3$. Then $\frac{dW}{dr}$ measures the rate at which the weight is increasing with respect to the radius.

Since $\qquad W = 0{\cdot}5r^3, \quad \frac{dW}{dr} = 3 \times 0{\cdot}5r^2 = 1{\cdot}5r^2.$

Consequently if the snowball is growing, the rate at which the weight is increasing is $1{\cdot}5r^2$ ozs. per inch increase of radius, when the radius is r inches. This rate depends on the size of r and increases as r increases. Thus when $r = 10$, the weight is increasing at the rate of $1{\cdot}5 \times 10^2 = 150$ ozs. per inch increase in the radius. This does not mean that the weight is 150 ozs. heavier when $r = 11$ than when $r = 10$, the increase in weight in that case would be $0{\cdot}5 \times 11^3 - 0{\cdot}5 \times 10^3 = 165{\cdot}5$ ozs. But it means that if r increases by a small amount, say from 10″ to 10·1″, i.e. by 0·1″,

the increase in the weight is approximately $0{\cdot}1 \times 150 = 15$ ozs. Actually it is $0{\cdot}5 \times 10{\cdot}1^3 - 0{\cdot}5 \times 10^3 = 15{\cdot}15$ ozs. And the smaller the increase is, the less will be the error per cent. in the increase of weight calculated at this rate.

Another illustration of the same idea may be useful. The shape of a hill-side is shown in Fig. 59 and the curve is represented by the equation $y = \frac{1}{1000}x^2$.

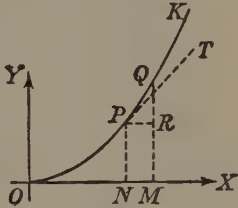

Fig. 59.

A man walking from O to P has moved x feet ($=ON$) horizontally and risen y feet ($=NP$) vertically. In climbing from P to Q he moves a further distance δx horizontally and δy vertically. For the portion of the hill PQ, he has climbed at the rate of δy feet vertically for δx feet horizontally or $\frac{\delta y}{\delta x}$ feet vertically per foot horizontally.

Now $\qquad y = \frac{1}{1000}x^2$ and $y + \delta y = \frac{1}{1000}(x + \delta x)^2$,

$$\therefore \frac{RQ}{PR} = \frac{\delta y}{\delta x} = \frac{\frac{1}{1000}[(x+\delta x)^2 - x^2]}{\delta x} = \frac{\frac{1}{1000}[2x \,.\, \delta x + (\delta x)^2]}{\delta x}$$

$$= \frac{1}{1000}[2x + \delta x].$$

His rate of climbing at the point P is the limit of the ratio $\frac{RQ}{PR}$ or $\frac{\delta y}{\delta x}$ when $\delta x \to 0$ and is what we have called $\frac{dy}{dx}$.

$\frac{dy}{dx}$ in fact measures the rate at which the vertical distance y is increasing with respect to the horizontal distance x.

In this case $\qquad \frac{dy}{dx} = \frac{2x}{1000} = \frac{x}{500}.$

Thus if $ON = 100$ feet, the rate at which he is climbing is $\frac{1}{5}$ foot vertically per foot horizontally. If the ground instead of sloping as the curve PQK does, sloped as the tangent line PT does, then he would continue to climb at the rate 1 in 5. But as it curves

above PT, the further he goes the greater is the rate at which he climbs: and the rate 1 in 5 is true only at the point P for which $ON = 100$.

Illustrations of the meaning and use of the Differential Coefficient

1. We have already seen that if $y = f(x)$ is the equation of a curve, $\frac{dy}{dx}$ = the rate at which y is increasing with respect to x = the gradient of the curve at any point = the tangent of the angle ψ which the tangent at that point makes with the x-axis.

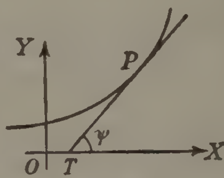

Fig. 60.

2. We have also seen that if a body is moving along a line so that it travels s feet in t seconds, where $s = f(t)$, then $\frac{ds}{dt}$ measures the rate at which s is increasing with respect to it, and ∴ = the velocity after t seconds.

3. Similarly if the velocity of the body is v feet per sec. after t seconds where $v = \phi(t)$, we denote its velocity after $t + \delta t$ seconds by $v + \delta v$ feet per sec., and so the velocity has increased by δv feet per sec. in δt seconds. The average rate of increase of the velocity is $\frac{\delta v}{\delta t}$ feet per sec. per sec.: and we say that the acceleration is $\frac{dv}{dt}$ ft./sec. per sec. after t seconds. It should be noted that the expression $\frac{dv}{dt}$

$$= \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}.$$

4. Similarly if a body is spinning about an axis and has rotated through an angle θ radians in t seconds where $\theta = f(t)$, it will turn through a further angle $\delta\theta$ radians in a further time δt seconds.

Its average angular velocity is $\frac{\delta\theta}{\delta t}$ radians per sec. and so its

actual angular velocity after t seconds is $\frac{d\theta}{dt}$. The symbol usually employed for angular velocity is ω and we write

$$\omega = \frac{d\theta}{dt}.$$

5. Fig. 61 is the graph of $y=f(x)$. The area between the curve and the lines CO, ON, NP is some function of x, if $ON=x$. Denote this area by A.

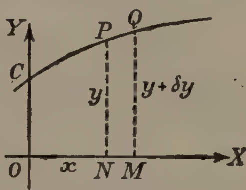

Fig. 61.

Then δA represents the area $PNMQ$ where $NM=\delta x$.

$\therefore$ δA lies between $NP.NM$ and $MQ.NM$, or between

$$y.\delta x \text{ and } (y+\delta y).\delta x;$$

$$\therefore \frac{\delta A}{\delta x} \text{ lies between } y \text{ and } y+\delta y.$$

$$\therefore \text{ in the limit when } \delta x \to 0,\ \frac{dA}{dx} = y = f(x).$$

This is a relation from which it is possible to calculate A. Geometrically it means that the rate at which the area A is increasing per unit increase of x is equal to the ordinate.

6. Let V be the volume of liquid in a vessel, x its depth and A the area of its surface.

A small increment δx in the depth will be produced by a small increment δV in the volume.

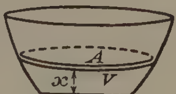

Fig. 62.

Now $\delta V = A\delta x$ approximately. (This relation would be exact if A were constant during the increase in volume.)

$\therefore$ $\frac{\delta V}{\delta x} = A$ approximately. The smaller δx is made the more nearly is the relation true. As δx decreases $\frac{\delta V}{\delta x}$ approaches its

limiting value which is called $\dfrac{dV}{dx}$. But $\dfrac{\delta V}{\delta x}$ is continually approaching A as δx decreases.

$$\therefore \frac{dV}{dx} = A.$$

Approximation

If y is a given function of x, we can calculate δy in terms of x and δx.

Since $\dfrac{dy}{dx}$ is the limiting value of $\dfrac{\delta y}{\delta x}$ when $\delta x \to 0$, we have $\dfrac{dy}{dx} = \dfrac{\delta y}{\delta x} + \epsilon$ where ϵ is a number which tends to 0 when δx tends to 0.

$$\therefore \frac{\delta y}{\delta x} \doteq \frac{dy}{dx} \text{ when } \delta x \text{ is small.}$$

Suppose y is the area of a square of side x.

Then $$y = x^2,$$

and $$\delta y = (x + \delta x)^2 - x^2$$
$$= 2x \,.\, \delta x + \overline{\delta x}|^2;$$
$$\therefore \frac{\delta y}{\delta x} = 2x + \overline{\delta x}|.$$

But $\dfrac{dy}{dx} = 2x$, $\therefore$ the error in replacing

$\dfrac{\delta y}{\delta x}$ by $\dfrac{dy}{dx}$ is (δx).

Hence to find δy approximately, we have

$$\frac{\delta y}{\delta x} \doteq \frac{dy}{dx} = 2x;$$
$$\therefore \delta y \doteq 2x \,.\, \delta x.$$

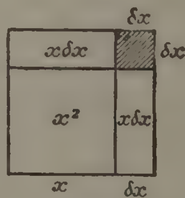

Fig. 63.

Fig. 63 shows that the error in taking δy as equal to $2x \,.\, \delta x$ is the shaded part $(\delta x)^2$.

Example 2.

Find the area of the ring between two concentric circles whose radii are 6″ and 6·1″.

Let A be the area and r the radius of the inner circle, then $A = \pi r^2$.

If a small increment δr is made in r, let the consequent increment in A be δA.

Now $\dfrac{\delta A}{\delta r}=\dfrac{dA}{dr}$ approximately when δr is small.

$$\therefore\ \delta A=\frac{dA}{dr}\,\delta r=2\pi r\,.\,\delta r.$$

$\therefore\ \delta A$ the area of the required ring

$$=2\pi\,(6)\,(\cdot1)=1\cdot2\pi=3\cdot8 \text{ sq. in. approx.}$$

The correct area of the ring is

$$\pi\,(6\cdot1)^2-\pi\,(6)^2=\pi\,(12\cdot1)\,(\cdot1)=1\cdot21\pi.$$

It will be noted that the value $2\pi r\,.\,\delta r$ is the area of a rectangle whose length is $2\pi r$ and width δr.

EXAMPLES V a

1. A stone falls s feet in t seconds where $s=16t^2$. Find $\dfrac{\delta s}{\delta t}$ in terms of t, δt. Interpret this result. What is the stone's velocity after $1\frac{1}{2}$ seconds?

2. The area A sq. cms. of a blot of ink is growing so that after t seconds $A=2t+\frac{1}{5}t^2$. What does $\dfrac{dA}{dt}$ represent and what is its value after 5 seconds? How much would A increase in the 6th second?

3. A stone thrown up into the air rises s feet in t seconds where $s=50t-16t^2$. What does $\dfrac{ds}{dt}$ represent? What is its value when $t=1$ and $t=2$? What is the meaning of a negative value of $\dfrac{ds}{dt}$?

4. Regarding the x-axis as sea-level, the slope of a hill is represented by the curve $y=350+\frac{1}{5}x-\frac{1}{1000}x^2$ for values of x from 0 to 500, the unit on each axis being one foot. What is the slope of the ground when $x=0$, 50, 100, 200? What is the meaning of a negative value of $\dfrac{dy}{dx}$?

5. If v cu. ins. of gas are under a pressure of p lbs. per sq. in., then if the temperature is constant, p and v are connected by the relation $pv=c$, where c is a constant. Prove that

$$\text{(i)}\ \frac{dp}{dv}=-\frac{c}{v^2},\qquad \text{(ii)}\ \frac{dv}{dp}=-\frac{c}{p^2},$$

and explain what each of these results means.

6. The relation between the vapour pressure p and the temperature θ of isopentane at constant volume is $p=b\theta-a$; deduce that the rate of change of the pressure with respect to the temperature is constant.

7. The electrical resistance R of a platinum wire at temperature θ is $R=R_0(1+a\theta+\beta\theta^2)$, where R_0, a, β are constants. Find an expression which gives the increase in resistance of the wire for a small change $\delta\theta$ in the temperature.

8. The coefficient of cubical expansion of a liquid at temperature θ is the rate of increase of volume per unit increase of temperature. If the volume of a gram of water at $\theta°$ C. is given by

$$V=1+8{\cdot}38\times10^{-6}(\theta-4)^2,$$

find the coefficient of cubical expansion at 0°.

9. If a man is standing on a cliff h feet high and if the angle of depression of a ship is $\theta°$ when its distance from the cliff is x feet, it can be proved that $\dfrac{dx}{d\theta}=-\dfrac{\pi h}{180\sin^2\theta}$. What does this mean? Illustrate your answer by taking the special case when $h=200$ and $\theta=30°$.

10. If the atmospheric pressure at a height of z feet above the earth's surface is p lbs. per sq. in., it can be proved that $\dfrac{dp}{dz}=-\dfrac{p}{c}$ where c is a constant whose value is approximately 26,000. State in words the meaning of this equation.

11. If a circle of radius a touches the x-axis at the origin O and if the tangent at any point P of the circle makes an angle ψ radians with Ox and if the length of the arc OP is s, then $\dfrac{ds}{d\psi}=a$. Interpret this result.

12. If a circular cylinder of volume V cu. ins. has height x inches and base radius r inches, prove that

(i) $\dfrac{dV}{dx}=\pi r^2$ if r is constant,

(ii) $\dfrac{dV}{dr}=2\pi rx$ if x is constant,

and explain the meaning of these two results.

13. Fig. 64 shows an arrangement for adjusting an electric lamp P. If the distances of P and W from the ceiling are y and z feet, it can be proved that $y+2z$ is constant. Prove that $\frac{dy}{dz}=-2$ and explain what this means.

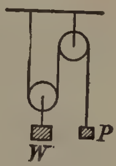

Fig. 64.

14. A circular cone has height x inches and semi-vertical angle a: its volume is V cu. ins. where

$$V=\tfrac{1}{3}\pi\tan^2 a \, . \, x^3.$$

What is $\frac{dV}{dx}$ if a is constant? Explain the result.

15. If a train of weight W tons is travelling v miles an hour, it can be stopped in x feet by a steady force of P tons where $P=\frac{Wv^2}{25x}$. Evaluate

(i) $\frac{dP}{dv}$ when W, x are constant,

(ii) $\frac{dP}{dx}$ when W, v are constant,

(iii) $\frac{dP}{dW}$ when v, x are constant,

and explain what each result means.

Express the laws given in Exs. 16—24 in symbols, using the notation for the differential coefficient:

16. The rate of change of the area A sq. ins. of a circle with respect to its radius r inches is proportional to r.

17. The rate of change of y per unit increase of x is c times the rate of increase of y per unit increase of z.

18. The velocity of a moving body which has travelled x feet in t seconds is proportional to x.

19. The rate of decrease in depth, x feet, of a liquid escaping from a bowl is proportional to $\sqrt{x}$.

20. The acceleration of a point moving in a straight line with velocity v feet per sec. is proportional to v^2.

21. The angular velocity of a rotating body is proportional to the angle θ turned through t seconds after starting.

22. The rate at which the number N of molecules in a gas is destroyed is proportional to the square of the number present.

23. The rate at which the temperature T° of a hot body in a room of constant temperature C° is falling is proportional to the excess of the temperature of the body above that of the room.

24. The gradient of a certain curve at any point is proportional to the abscissa of that point.

25. The volume of a sphere of radius r inches is $\frac{4}{3}\pi r^3$ cu. ins. Find the approximate increase of volume of a sphere when the radius increases from 2 to 2·01 inches.

26. Fig. 65 represents the graph of $y=3x$; $ON=x$ and $A=$ area of $\Delta\, ONP$. Evaluate $\delta A-\frac{dA}{dx}\times\delta x$ in terms of δx, and illustrate the difference geometrically.

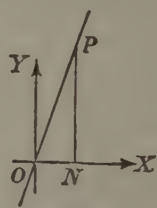

Fig. 65.

27. If $y=x^3$, evaluate $\delta y-\frac{dy}{dx}\times\delta x$, when $x=1$, $\delta x=h$. Draw the graph of $y=x^3$ and represent this difference geometrically.

28. What is the approximate value of

(i) $(12{\cdot}03)^4-12^4$, (ii) $(10{\cdot}02)^5$?

29. If $x=3y^2$, $y=2z-1$, find an approximate expression for δx in terms of z, δz.

30. If $x=y^3+5$, $y=z^2-7z-3$, find an approximate expression for δx in terms of z, δz.

Functions of a function

It often happens that we require to find the rate of change of a function y with respect to a function x when we are given y as a function of another variable z and also z as a function of x. y is then said to be a function of a function.

Suppose for example $y=z^2$ where $z=2+3x^2$. If there is a small increase δx in x, the corresponding increase δz in z is given by

$$\delta z=\{2+3(x+\delta x)^2\}-\{2+3x^2\},$$

and approximately

$$\delta z = \frac{d}{dx}(2+3x^2) \times \delta x \quad \text{or} \quad 6x \, . \, \delta x.$$

The increase δz in z causes an increase δy in y given by

$$\delta y = (z+\delta z)^2 - z^2,$$

and approximately $\quad \delta y = \dfrac{d}{dz}(z^2) \times \delta z = 2z \, . \, \delta z.$

$\therefore$ approximately

$$\begin{aligned} \delta y &= 2z \, . \, \delta z \\ &= 2(2+3x^2)\, 6x \, . \, \delta x. \end{aligned}$$

$\therefore$ the approximate value of $\dfrac{\delta y}{\delta x}$ is $12x(2+3x^2)$,

and in the limit when $\delta x \to 0$ we have the exact relation

$$\frac{dy}{dx} = 12x(2+3x^2).$$

It is not obvious that the errors made in the approximations do not affect the limiting value.

We can, however, prove that the final result obtained is exact and not approximate as follows:

$$\frac{\delta z}{\delta x} = \frac{dz}{dx} + \epsilon_1,$$

where $\epsilon_1 \to 0$ when $\delta x \to 0$,

$$\frac{\delta y}{\delta z} = \frac{dy}{dz} + \epsilon_2,$$

where $\epsilon_2 \to 0$ when $\delta z \to 0$, i.e. when $\delta x \to 0$.

$$\therefore \frac{\delta y}{\delta z} \times \frac{\delta z}{\delta x} = \left(\frac{dz}{dx} + \epsilon_1\right)\left(\frac{dy}{dz} + \epsilon_2\right),$$

$$\therefore \frac{\delta y}{\delta x} = \frac{dy}{dz} \cdot \frac{dz}{dx} + \epsilon_1 \frac{dy}{dz} + \epsilon_2 \frac{dz}{dx} + \epsilon_1 \epsilon_2.$$

But ϵ_1 and ϵ_2 can be made as small as we please by making δx sufficiently small.

$\therefore$ in the limit, when $\delta x \to 0$,

$$\underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx},$$

and so

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Thus in the example taken above

$$y = z^2, \qquad z = 2 + 3x^2,$$

$$\frac{dy}{dz} = 2z \text{ and } \frac{dz}{dx} = 6x.$$

$$\therefore \frac{dy}{dx} = 2z \times 6x = 12x(2 + 3x^2).$$

In this special case, the result can be checked as follows:

$$y = z^2 = (2 + 3x^2)^2 = 4 + 12x^2 + 9x^4.$$

$$\therefore \frac{dy}{dx} = 24x + 36x^3 = 12x(2 + 3x^2).$$

The idea expressed in the statement $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ is similar to that used when we are finding A's rate of running compared with C's rate, given A's rate compared with B's rate and B's rate compared with C's rate.

Thus if A travels twice as fast as B and B travels 3 times as fast as C, we know that A travels 2×3 times as fast as C.

This theorem may be used for differentiating x^n when n is fractional or negative.

The method is illustrated in the following special cases; the proof for the general case is given in Part II.

Example 3.

Find $\frac{dy}{dx}$ when $y = x^{\frac{2}{3}}$.

We have $\quad y^3 = x^2, \quad \therefore \frac{d}{dx}(y^3) = 2x.$

Now
$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \times \frac{dy}{dx}$$

$$= 3y^2 \times \frac{dy}{dx}.$$

$$\therefore 3y^2 \times \frac{dy}{dx} = 2x.$$

$$\therefore \frac{dy}{dx} = \frac{2x}{3y^2} = \frac{2x}{3x^{\frac{4}{3}}} = \tfrac{2}{3}x^{-\frac{1}{3}}.$$

Example 4.

Find $\dfrac{dy}{dx}$ when $y=x^{-3}=\dfrac{1}{x^3}$.

Let $\dfrac{1}{x}=z$, then $y=z^3$.

Now
$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$
$$=(3z^2)\times\left(-\frac{1}{x^2}\right) \qquad \left[\text{since } \frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2} \text{ (see p. 29)}\right]$$
$$=3\,.\,\frac{1}{x^2}\,.\left(-\frac{1}{x^2}\right)=-3x^{-4}.$$

It will be seen that these results obey the rule given for differentiating x^n when n is a positive integer.

A further use of the principle is illustrated in the following example:

Example 5.

Find $\dfrac{dy}{dx}$ when $y=\sqrt{x^2+1}$.

Let $z=x^2+1$, then $y=z^{\frac{1}{2}}$.

Hence
$$\frac{dz}{dx}=2x \text{ and } \frac{dy}{dz}=\tfrac{1}{2}z^{-\frac{1}{2}}.$$
$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$
$$=2x\times\tfrac{1}{2}z^{-\frac{1}{2}}$$
$$=x\times\frac{1}{\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}.$$

EXAMPLES V b

1. Find, by using the formula for differentiating a function of a function, the values of the derivatives of

(1) $x^{\frac{1}{3}}$; (2) $x^{\frac{3}{4}}$; (3) x^{-2}; (4) x^{-4}; (5) $x^{-\frac{1}{2}}$.

2. Evaluate $\dfrac{dp}{dv}$ when $pv^{1\cdot 4}=c$ (a constant).

3. If $F=\dfrac{K}{x^2}$, write down $\dfrac{dF}{dx}$ and $\dfrac{d^2F}{dx^2}$.

4. (i) If $V=\frac{4}{3}\pi r^3$ and $s=4\pi r^2$, find $\frac{ds}{dV}$.

(ii) If $s=k\,.\,V^{\frac{2}{3}}$, find $\frac{ds}{dV}$.

5. Differentiate with respect to x:

(i) $\frac{1}{2x+3}$; (ii) $\frac{1}{4-x^2}$;

(iii) $\frac{1}{(5-3x)^2}$; (iv) $\frac{5}{(x-1)(2-x)}$.

6. Differentiate with respect to x:

(1) $\sqrt{1+x}$; (2) $\sqrt{1-x^2}$; (3) $(3+4x)^{\frac{1}{2}}$;

(4) $(1-x^2)^{\frac{3}{2}}$; (5) $(1-x^2)^{-1}$; (6) $(x^2-3x+7)^{\frac{1}{3}}$.

7. Prove that if $y=\sqrt{1+x^2}$ then $y\frac{dy}{dx}=x$.

8. What functions have as their derivatives:

(1) $x^{\frac{1}{2}}$; (2) $3x^2$; (3) $x^{-\frac{1}{2}}$; (4) t^{-2}; (5) $t^{\frac{2}{3}}+\frac{2}{3}t^{-\frac{1}{3}}$?

Example 6.

A filter paper is in the form of a cone, base-radius 2″, altitude 3″. If water is flowing out at the bottom at a constant rate of 5 cu. ins. per min., find the rate at which the level of the liquid is falling when the depth is 2″.

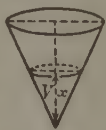

Fig. 66.

Let the depth of the liquid after t minutes be x inches and let its volume be V cu. ins. Then the radius of the surface of the liquid is $\frac{2x}{3}$ inches.

$$\therefore\ V=\tfrac{1}{3}\pi\left(\tfrac{2}{3}x\right)^2.\,x=\tfrac{4}{27}\pi x^3.$$

Now for a small interval of time δt, the increase in volume δV equals approximately $-5\delta t$ cu. ins. (the sign is $-$ because the volume is *diminishing*). Or we can say without any error at all

$$\frac{dV}{dt}=-5.$$

It is required to find the rate of decrease of x.

Now $$\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dt}.$$

But $$V=\frac{4}{27}\pi x^3, \quad \therefore \frac{dV}{dx}=\frac{4}{9}\pi x^2.$$

$$\therefore -5=\frac{4}{9}\pi x^2 \times \frac{dx}{dt}.$$

$$\therefore \frac{dx}{dt}=-\frac{45}{4\pi x^2}.$$

$\therefore$ when $x=2$, $$\frac{dx}{dt}=-\frac{45}{4\pi\times 4}=-\frac{45}{16\pi} \text{ ins. per min.}$$

$$=-0{\cdot}895 \text{ ins. per min.}$$

$\therefore$ the rate at which the level is falling is $+0{\cdot}895$ ins. per min. when the depth is 2 ins.

It is important to notice the necessity of considering the volume for a depth of x inches, in spite of the fact that it was only required to calculate the rate of fall when $x=2$. Unless V is expressed in terms of a variable, we cannot calculate its rate of change, as is necessary in this example.

Of course when $x=2$, $V=\frac{4}{27}\pi\times 2^3=\frac{32\pi}{27}$ cu. ins. But from the relation $V=\frac{32\pi}{27}$, it is impossible to make any calculation about the *rate of change* of V.

Example 7.

Show that $\frac{d}{dx}\left(\frac{v^2}{2}\right)=\frac{d^2x}{dt^2}$, where x represents the distance travelled in time t by a body moving with velocity v at time t.

Now $$\frac{dx}{dt}=v.$$

$$\therefore \frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{dv}{dt}.$$

$\therefore$ by the preceding theorem we have

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}=\frac{dv}{dx}\times v.$$

Now $$\frac{d}{dx}\left(\frac{v^2}{2}\right)=\frac{d}{dv}\left(\frac{v^2}{2}\right)\times\frac{dv}{dx}$$

$$=v\times\frac{dv}{dx}.$$

$$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right)=\frac{d^2x}{dt^2}.$$

This example is of great importance in Dynamics.

EXAMPLES V c

1. A long trench is being dug by a party of soldiers who remove x cu. feet of soil in t minutes, where $x = 12t - \frac{1}{40}t^2$; at what rate is the soil being removed at the end of half an hour?

2. A wheel is made to rotate about its axis with uniform angular acceleration so that after t minutes it has rotated through $12t^2$ degrees. When will its speed attain the rate of 50 revolutions per hour?

3. The temperature of a cube is rising so that each edge expands at the rate of 0·003 cm. per min. At what rate is the volume increasing when the edge is 10 cms.?

4. A rectangle has a fixed perimeter of 20″. Find the rate of increase of the area with respect to the breadth when the breadth is 2″.

5. A circular blot on a sheet of blotting paper is supplied with ink so that its radius grows at the rate of 0·1 in. per sec. At what rate is the area of the blot increasing when the radius is 1 in.?

6. A certain mass of gas is contained in a vessel of v cu. ins. under a pressure of p lbs. per sq. in., where $pv = 1200$. If the volume increases at the rate of 40 cu. ins. per min., find the rate of change of the pressure when the volume is 20 cu. ins.

7. A wheel is set spinning and turns through θ° in t seconds, where $\theta = 120t - 6t^2$: at how many revolutions per minute is it rotating after 6 seconds and when does it come to rest?

8. A glass is so constructed that when the depth of fluid in it is y ins., the volume is V cu. ins. where $V = y^3 - \frac{1}{4}y^4$. Water is poured into it at the rate of 1 cu. in. per sec.: at what rate is the level rising when the depth is (i) 2 inches, (ii) h inches? Explain the answers obtained when $h = 3$ and $h = 3·5$.

9. The section of a trough 12 feet long is an isosceles triangle of height 20 ins. and base (upwards) 12 ins.; water is poured into it at the rate of 2 cu. feet per min. Find the rate at which the level is rising after 3 minutes.

10. Corn is being thrown out of a shoot at the rate of 3 cu. feet per min. and is forming a heap on the ground in the shape of a circular cone of vertical angle 90°. At what rate in inches per sec. is the altitude increasing after 4 minutes?

11. A sphere is expanding so that its volume increases at the rate of 4 cu. in. per min.: at what rate is the area of its surface increasing when its radius is 2 inches?

12. A vessel has the form of an inverted circular cone whose height is equal to its base-diameter. Water is poured steadily into it at the rate of 1 cu. foot in 5 minutes. Find the speed in inches per min. at which the water level is rising after 2 minutes.

13. The contents V cu. ins. and the depth x inches of water in a vessel are connected by the relation $V=5x^2-\frac{1}{6}x^3$. If the depth is 4 inches, what is the area of the upper surface of the water?

14. A beam AB of uniform section is such that the weight of any portion AP is $\frac{1}{10}x^3$ lbs. where $AP=x$ feet. If C is the mid-point of AB, compare the densities of the material at C and B.

15. The density of the material of which a sphere is composed is a function $f(r)$ of the distance r inches from the centre. The weight of such a sphere of radius x inches is $2x^3-0{\cdot}1x^4$ lbs. Find $f(r)$.

16. If I is the intensity of light after passing through a sheet of glass of thickness x, then $\frac{dI}{dx}=-kI$ where k is a constant: explain in words what this means.

17. A body moves so that its velocity v varies as the square of the distance S it has travelled; prove that its acceleration $\frac{dv}{dt}$ varies as S^3.

18. Find the possible error in the area of a circle whose circumference is measured and found to be 48″ with a possible error of $\frac{1}{50}$″.

19. A point moves along the curve $y=x^4$ so that its velocity parallel to OX is always 2 ft./sec. What is its velocity parallel to OY when

(1) $x=3$; (2) $y=0$; (3) $x=-1$?

20. Given $c^2=a^2+b^2-2ab\cos C$, find a formula for the percentage error in c due to an error of 2 °/。 in the value of a, the other terms being supposed correct.

REVISION PAPERS 1—5

R. 1

1. Draw a rough graph of $y=\frac{x^2}{1-x^2}$.

What lines are its asymptotes?

2. For the curve $y=x-2x^2$, what is $\frac{\delta y}{\delta x}$ when $x=1$ if $\delta x=1$, 0·1, 0·001? To what limit does $\frac{\delta y}{\delta x}$ approach as $\delta x \to 0$?

3. If the relation between the pressure p lbs./sq. ft. and volume v cu. ft. of a gas is $pv=200$, express δp in terms of v and δv. What is the change in pressure if the volume expands from 20 to 21 cu. ft.?

4. Calculate the values of $\frac{x^2-9}{x-3}$ when $x=2·99$ and when $x=3·01$. What is the limiting value of this fraction when $x \to 3$?

5. Find the limiting value of $\frac{10x+3}{5x+8}$ as x tends to infinity.

6. A conical wine-glass whose semi-vertical angle is 30° contains liquid to a depth x:

(1) Find the area of the surface of the liquid in terms of x.

(2) If the volume of liquid is V, find δV in terms of x and δx and find

$$\operatorname*{Lt}_{\delta x \to 0} \frac{\delta V}{\delta x}.$$

Account for the results of (1) and (2) being the same.

R. 2

1. Find from first principles the gradient of the curve $y=x^2+3x-2$.

2. Differentiate with respect to x: $(1-x)^3$; $(2+x)(3+x^2)$; $x+\frac{2}{x}$.

3. At what points of the curve $y=2x^3-12x^2+10$ are the tangents parallel to OX?

4. Find the equation of the tangent and of the normal to

$$2y=x^2-3x+4$$

at the point (1, 1).

5. From the following table show as nearly as you can that if $y=e^x$ then $\frac{dy}{dx}=y$.

x	2	2·01	3	3·01	4	4·01
e^x	7·389	7·463	20·09	20·29	54·60	55·14

6. Find $\frac{d^2s}{dt^2}$ if $s=6t+16t^2$.

R. 3

1. If Fig. 22, p. 15, represents part of the graph of $y=1+x^2+\frac{1}{2}x^3$ and if $ON=2$, $NM=1$, what is the length of QR?

2. If the abscissae of three points P, Q, R on $y=3x^2$ are x, $x-h$, $x+h$, show that the gradient of the chord QR is the same as the gradient at P.

3. Differentiate with respect to x:

(i) $5-x$; (ii) $\frac{3x^2-1}{x}$; (iii) $(3x^2-1)^2$.

4. An open box made of thin cardboard has two square sides and is fitted with a cardboard lid which covers the top, the front and the two square sides. What is the maximum volume of the box if the total area of the cardboard is 3 sq. feet?

5. If $y=\frac{x+3}{1-x}$, find an approximate expression for δy in terms of x, δx.

6. If $x=5t-3$ and $y=2-3t$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$.

R. 4

1. Find the gradient at any point of $y=x+\frac{4}{x}$ and evaluate it when $x=-4, -2, -0{\cdot}1, 1, 2, 4$. Sketch the graph.

2. Find the equation of the normal to $y=x+\frac{4}{x}$ at the point (2, 4).

3. Find the turning points of the function $4x^3-27x+5$ and determine whether the function is a maximum or a minimum at these points.

4. When a rod 10 inches long swings like a pendulum, the tendency to break at a point h inches below the point of suspension varies as $h(10-h)^2$. Find where it is most likely to break.

5. The speed of a body after moving in a straight line for t seconds is $(12t-3t^2)$ ft./sec. Find how far it goes in the third second.

6. Show that $x^3-3x^2+6x+10$ has no maximum nor minimum. Illustrate by a graph. Is there a point of inflexion?

R. 5

1. State with the notation of the calculus

(1) the condition that the gradient at any point $P(x, y)$ of a curve equals the gradient of the line joining P to the origin;

(2) the rate at which a sum of money P is increasing after t years equals the rate of increase per year at r °/ₒ per annum.

2. Put the following symbolical statement into words:

$$E=CR+L\frac{dC}{dt},$$

where E is the electromotive force in a circuit, C the current, R the resistance and L the self-inductance.

3. Sketch the graph of $y=x^2(x-2)(x-5)$ and of its first and second derived curves.

4. Fig. 67 represents a curve called the Cissoid of Diocles; its equation is

$$y=\sqrt{\left\{\frac{(4-x)^3}{x}\right\}}.$$

Find the maximum area of the triangle ONP.

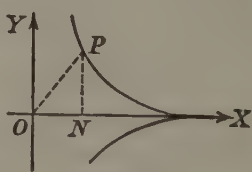

Fig. 67.

5. $P(h, k)$ is a point on $y=cx^n$ (c being constant). If the tangent at P meets OX in T, prove that $OT=\frac{h(n-1)}{n}$.

6. What functions have as their derivatives:

(i) $3x^3-4x^4$; (ii) $x^2-\frac{1}{x^2}$; (iii) $(3x-2)^2$; (iv) $\sqrt{(2x)}$?

MISCELLANEOUS EXAMPLES 1—6

M. 1

1. The distance s feet travelled by a body in t seconds from a fixed point O is given by the following table:

s	0	·5	2	4·5	8	12·5
t	0	1	2	3	4	5

Plot the graph showing the relation between s and t and from it find the speed of the body when $t=3$.

2. Air is blown into a spherical ball so that its radius r feet increases at the rate of c ft./sec. Find an expression giving the rate at which the air enters. [Volume of sphere$=\frac{4}{3}\pi r^3$.]

3. A closed vessel tapers to a point both at its top E and its bottom F and is fixed with FE vertical. When the depth of the liquid in it is x inches, the volume of the contents is $x^2(15-x)$ cu. ins. What is the length of EF?

4. Find the gradient of the curve $y=x(x-1)(x-2)$ at the points where it cuts the X-axis. Sketch the curve.

5. If the radius of a sphere increases 1 °/。 owing to an increase of temperature, find the percentage increase in the surface. [$S=4\pi r^2$.]

M. 2

1. The area of a square is to be 400 sq. feet. What error in the length of a side will make the area 1 sq. foot too small?

2. A body is falling so that its velocity (ft./sec.) is altering according to the law $\frac{dv}{dt}=32-\frac{v^2}{200}$. Find the greatest velocity it acquires.

3. The radius (r) of a cylinder is increasing while the height remains constant. If the growth in volume during time t is proportional to t^2, find an expression for the rate of growth of the radius, and prove that it is proportional to $\frac{t}{r}$.

4. Water is poured into a vessel of circular cross-section and the corresponding readings of the volume and depth are given by :

d	1	2	3	4	inches
V	1·05	8·38	28·27	67·01	cu. inches

Plot these results and find from your graph the area of the cross-section at each depth. Sketch the shape of the vessel.

5. A wire fixed at two points A and B in a horizontal line carries a heavy weight at its mid-point O. When the length of the wire is $2s$ feet the depth of O below AB is y. When the length alters to $2(s+h)$ feet the depth is $y+k$ feet. Find the relation between h and k and deduce an approximation for $\frac{k}{h}$ when h is very small. Find $\frac{dy}{ds}$ in terms of s, y. (Army.)

M. 3

1. The path of a projectile is given by the equation

$$y = x - \frac{x^2}{64}.$$

Find the angle at which it was projected and find the direction in which it is moving when $x=4$.

2. The distance s feet which a body has travelled in t seconds is given by $s=2t^3-9t^2+12t+6$. Find when its acceleration is zero, and find its velocity at that time.

3. If an aeroplane of weight W tons is moving horizontally at V feet per sec. under horse-power H, then $H=\frac{aW^2}{V}+bV^3$ where a, b are constants. For what speed is the horse-power least?

4. The relation between the pressure p, the volume v, and the temperature t of a gas is $pv=B(t+273)$. Find the relation connecting simultaneous small changes δp, δv, δt in p, v, and t. (Neglect $\delta p \,.\, \delta v$.)

The following table gives the readings of an experiment:

Time	Pressure (atmospheres)	Volume c.c.	Temperature
9 hr. 14 min.	5·75	23·4	—
9 hr. 15 min.	5·82	23·9	27·0° C.
9 hr. 16 min.	5·87	24·2	—

Find the value of B. If the rate of increase of pressure and volume in the interval 9 hr. 14 min.—9 hr. 16 min. represents the rate of increase at 9 hr. 15 min., find in degrees/min. the rate of increase of the temperature at 9 hr. 15 min. (Army.)

5. The equation of the curve taken by a telegraph wire between two posts at the same level, 60 m. apart, the sag being 30 cms., is $2c(y-c)=x^2$, the axis of y being a line parallel to and mid-way between the posts. Find c and the slope of the wire at the ends of the span, the unit on each axis being 1 cm. (Army.)

M. 4

1. A block of ice in the form of a cube, whose edge is 1 metre, is melting so that its volume decreases at a uniform rate, the block remaining cubical. If the rate of melting is such that the edge measures 50 cms. after 25 hours, find its length after 5, 10, 20 hours. Find the rate at which the edge is decreasing at the end of 15 hours.

2. A semi-circle AKB is described on AB as diameter. C is a point in AB and shaded semi-circles are described on AC and CB as diameters both being inside AKB. If $AB=2r$ cms., find the positions of C for which the unshaded area is a maximum. If C starts from A and moves at the rate of 1 cm./sec., find if $r=20$ the rate at which the unshaded part is increasing when AC is 6 cms.

3. A hemispherical basin of radius 6″ contains water which runs out at the rate of 2 cu. in./sec. The volume of the cap of a sphere of radius r whose height is x being $\frac{\pi x^2}{3}(3r-x)$, find the rate at which the depth of water is decreasing (1) when the basin is full, (2) when the depth is 4″.

4. A lamp is 40 feet above the ground and a stone is dropped from the same height at a point 10 feet away. Find the speed of the shadow on the ground (1) after 1 second, (2) when the stone has fallen 9 feet. [In t seconds the stone falls $16t^2$ feet.]

5. Without drawing a graph find between what values of x the function $1+12x-x^3$ is increasing.

M. 5

1. Find the coordinates of the points where $y^2=16x$ and $y=2x-6$ intersect. Find the equations of the tangents to the parabola at these points.

2. The ordinate of the parabola $y=x^2$ moves along the axis at a uniform speed of 1″ per sec. Find the rate at which the ordinate is growing when it is 8″ from the origin. [Unit on each axis 1″.]

3. A V-shaped trough is 10 feet long, its cross-section is 3 sq. feet and its depth 2 feet. If it is full of water and empties at the rate of 10 cu. feet per min., find the rate at which the depth begins to diminish.

4. Show that the Dynamical equation $\dfrac{d^2x}{dt^2}=-kx$ is equivalent to

$$\frac{d}{dx}\left(\frac{v^2}{2}\right)=-kx.$$

Show that the relation $v^2=-kx^2+c$ where c is a constant leads to the given equation.

When one end of a flexible bar is fixed and the other end P pulled aside and then let go, P will describe approximately a straight line and will oscillate about its position of rest. The acceleration of P being given by the equation $\dfrac{d^2x}{dt^2}=-\dfrac{x}{9}$, where x'' is the distance of P from its mean position, at any time t sec., find the speed of P when passing through its mean position after being drawn aside 2″.

5. Find the equation of the tangent at (x_1, y_1) to the curve $xy=c$. Find the coordinates of A and B where the tangent cuts the axes and prove that AOB is of constant area.

M. 6

1. If in Fig. 67, OP is the line $y=x$, find the area of the triangle OPN.

2. The graph of the path of a shell fired from the origin O is

$$y=\frac{3x}{4}-\frac{x^2}{25600},$$

where OX is the ground level and the unit on each axis is 1 foot. At what angle with the ground is the shell moving when it has travelled a horizontal distance of 2000 yards and at what height is it when moving horizontally?

3. A particle moves in a straight line, starting with a velocity of 10 feet per sec. and after 1 second having a velocity of 4 feet per sec. If the relation between the distance s feet travelled after t seconds is $s=at+bt^3$ where a, b are constants, find its velocity after 2 seconds.

4. The angle of a sector of a circle of radius 1 foot is x right angles ($x<4$). The sector is folded to form a circular cone; prove that the volume of the cone is $\frac{\pi x^2}{192}\sqrt{(16-x^2)}$ cu. feet, and find the value of x for which this is greatest.

5. A wash-tub is 10 inches deep and is so constructed that when the depth of water in it is x inches, the volume of the water is

$$x(480+12x+\tfrac{1}{10}x^2) \text{ cu. inches};$$

(i) what is the area of the base of the tub?

(ii) if water is poured in at the rate of 5 cu. ins. per sec., at what rate is the level rising when the depth is 5 inches?

CHAPTER VI

INDEFINITE INTEGRALS

In the previous chapters we have discussed various rules for differentiating certain simple functions, i.e. for solving the problem: given $y = f(x)$, find $\frac{dy}{dx}$. We shall now consider the reverse operation: given $\frac{dy}{dx}$, express y as a function of x.

This process is called **Integration**.

Suppose for example $\frac{dy}{dx} = 6x$, what is y? We know that $\frac{d}{dx}(3x^2) = 6x$ and $\therefore$ $3x^2$ is certainly *one* possible value of y.

But $\frac{d}{dx}(3x^2 + c)$ also equals $6x$, if c is any constant, $\therefore$ $y = 3x^2 + c$ is also a possible answer.

We can prove that this is the most general answer as follows.

Let $y = f(x)$ be the general solution of $\frac{dy}{dx} = 6x$,

$$\therefore \frac{d}{dx} f(x) = 6x: \text{ but } \frac{d}{dx}(3x^2) = 6x,$$

$$\therefore \frac{d}{dx}[f(x) - 3x^2] = 0.$$

Now the statement that $\frac{d}{dx}[f(x) - 3x^2] = 0$ means that as x changes the function $f(x) - 3x^2$ never changes, or in other words is a constant,

$$\therefore f(x) - 3x^2 = c, \text{ a constant,}$$

$$\therefore f(x) = 3x^2 + c.$$

This argument shows that if $y = f(x)$ is one possible answer to

the equation $\frac{dy}{dx} = \phi(x)$, then $y = f(x) + c$ is the most general answer, where c is an *arbitrary constant*, that is to say any number chosen at will. It is useful to consider the geometrical meaning of this result.

If $\frac{dy}{dx} = 6x$ leads to $y = f(x)$, the graph of $y = f(x)$ is such that at every point its gradient equals $6x$ and so the *direction* of the tangent corresponding to any value of x is known. We can take as starting-point for example any point whatever on OY, but as soon as the starting-point is fixed the graph is completely determined. In other words, the graph is any one of a series of "parallel" curves and in this case the arbitrary constant c in $y = 3x^2 + c$ is the length cut off on the Y-axis by the particular curve selected.

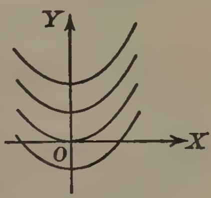

Fig. 68.

The following example from dynamics offers another illustration of the fact that the arbitrary constant is due to insufficient data.

A body is moving so that its velocity is v feet per sec. after t secs. and its acceleration is 3 ft./sec. per sec.: find v in terms of t.

Its acceleration is $\frac{dv}{dt}$

$$\therefore \frac{dv}{dt} = 3,$$

$$\therefore v = 3t + c,$$

where c is an arbitrary constant.

Now we are told at what rate the velocity is increasing but we are not told with what velocity it starts: and therefore an arbitrary constant comes in which can be found if for example we are told the initial velocity.

In fact in the equation $v = 3t + c$ we see that $v = c$ when $t = 0$; and therefore the arbitrary constant c is actually the initial velocity.

Notation

It is convenient to have a symbol for this reverse process.

If $\frac{d}{dx}[f(x)] = \phi(x)$, we say that

$$f(x) = \int \phi(x) \, . \, dx,$$

and read it as "integral $\phi(x) \, . \, dx$."

For example, $$\frac{d}{dx}(x^3 - 4x^2) = 3x^2 - 8x,$$

$$\therefore \int (3x^2 - 8x)\, dx = x^3 - 4x^2 + c.$$

We have proved that when n is a positive integer

$$\frac{d}{dx}(x^{n+1}) = (n+1)\, x^n,$$

and it will be proved later that this relation is true for all values of n; hence we have the general result

$$\int x^n \, . \, dx = \frac{1}{n+1} x^{n+1} + c.$$

Any equation which involves $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ etc. is called a *differential equation* and the process of determining y as a function of x is called *solving* or *integrating* the equation.

Example 1.

Find $$\int (3x^4 - 7x^2 + x + 2)\, dx.$$

We have to find a function $f(x)$ such that

$$\frac{d}{dx}[f(x)] = 3x^4 - 7x^2 + x + 2.$$

Now $$\frac{d}{dx} x^5 = 5x^4, \quad \therefore \frac{d}{dx}(\tfrac{3}{5}x^5) = 3x^4,$$

and $$\frac{d}{dx}(-\tfrac{7}{3}x^3) = -7x^2, \quad \frac{d}{dx}(\tfrac{1}{2}x^2) = x, \quad \frac{d}{dx}(2x) = 2,$$

$$\therefore \frac{d}{dx}(\tfrac{3}{5}x^5 - \tfrac{7}{3}x^3 + \tfrac{1}{2}x^2 + 2x) = 3x^4 - 7x^2 + x + 2,$$

$$\therefore f(x) = \tfrac{3}{5}x^5 - \tfrac{7}{3}x^3 + \tfrac{1}{2}x^2 + 2x + c.$$

The process may be written more shortly as follows:

The given integral $= 3\int x^4\, dx - 7\int x^2\, dx + \int x\, dx + \int 2\, dx$

$$= 3 \times \frac{x^5}{5} - 7 \times \frac{x^3}{3} + \frac{x^2}{2} + 2x + c.$$

Example 2.

Find $\int \frac{dx}{x^3}$ and $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) dx$.

(i) $\frac{d}{dx}\left(\frac{1}{x^2}\right)=\frac{d}{dx}x^{-2}=-2x^{-3}=-\frac{2}{x^3}$,

$$\therefore \int \frac{dx}{x^3}=-\frac{1}{2x^2}+c.$$

(ii) $\frac{d}{dx}x^{\frac{3}{2}}=\frac{3}{2}x^{\frac{1}{2}}$, $\quad \therefore \int \sqrt{x}\,dx=\frac{2}{3}x^{\frac{3}{2}}$.

$\frac{d}{dx}x^{\frac{1}{2}}=\frac{1}{2}x^{-\frac{1}{2}}$, $\quad \therefore \int \frac{1}{\sqrt{x}}dx=2x^{\frac{1}{2}}$,

$$\therefore \text{integral}=\tfrac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c.$$

Example 3.

A body moves in a straight line OA so that its velocity after t seconds is $12t-t^2$ feet per sec. Its distance from O after 3 secs. is 60 feet. Find its distance from O after t secs.

Suppose it is at a distance of s feet from O after t secs. Then its speed $=\frac{ds}{dt}=12t-t^2$; $\quad \therefore s=6t^2-\frac{1}{3}t^3+c.$

But when $t=3$, $s=60$, $\quad \therefore 60=6\times 3^2-\frac{1}{3}\times 3^3+c$,

$\therefore c=60-54+9=15$, $\quad \therefore s=6t^2-\frac{1}{3}t^3+15.$

EXAMPLES VI

1. Prove that $y=4x^3+3$ is a solution of the differential equation $\frac{dy}{dx}=12x^2$. What is the general solution?

2. Prove that $y=x-\frac{1}{x}$ is a solution of $\frac{dy}{dx}=1+\frac{1}{x^2}$. What is the general solution?

3. Prove that $y=2\sqrt{x}+\frac{3}{4}$ is a solution of $\frac{dy}{dx}=\frac{1}{\sqrt{x}}$. Find the solution if $y=3$ when $x=1$.

4. Find y in terms of x if

(i) $\frac{dy}{dx}=x^2$ and $y=1$ when $x=1$.

(ii) $\frac{dy}{dx}=x^3-4x$ and $y=2$ when $x=0$.

(iii) $\frac{dy}{dx}=(2x-1)^2$ and $y=1$ when $x=-1$.

5. Integrate with respect to x the following expressions:

(i) x. (ii) x^2+2. (iii) $7x^2$. (iv) $4x^3$. (v) $2x^5$.
(vi) $7x^{10}$. (vii) $5x^2-3x+8$. (viii) $(4x+1)^2$. (ix) $(x-1)(x+2)$.
(x) x^{-2}. (xi) x^{-3}. (xii) $\frac{5}{x^4}$. (xiii) $\frac{7}{x^{10}}$. (xiv) $x^2+\frac{1}{x^2}$.
(xv) $x^{\frac{1}{2}}$. (xvi) $\sqrt[3]{x}$. (xvii) $x^{-\frac{1}{2}}$. (xviii) $\frac{1}{\sqrt[3]{x}}$. (xix) $5x^{\frac{3}{2}}$.
(xx) $(3x-9)^{10}$. (xxi) $\sqrt{(7x+1)}$. (xxii) $\frac{1}{(5x-2)^3}$.

6. Evaluate the following expressions:

(i) $\int(x^2-3x+4)\,dx$. (ii) $\int\frac{dx}{x^5}$. (iii) $\int x\sqrt{x}\,dx$.
(iv) $\int x^{-1\cdot 3}\,dx$. (v) $\int\frac{2x-1}{x^3}\,dx$. (vi) $\int\frac{dx}{x^{0\cdot 7}}$.
(vii) $\int 3dx$. (viii) $\int(4x-1)^5\,dx$. (ix) $\int(3x^{-2}-4x^{\frac{1}{2}})\,dx$.

7. The gradient of a graph is given by the equation $\frac{dy}{dx}=12x^2-7$: if the graph cuts Oy at a distance of 3 units from O and above O, express y in terms of x. Prove that the graph cuts Ox where $x=1$ and find where else it cuts Ox.

8. A body moves along a line OA so that its velocity t seconds after passing O is $3+5t$ feet per sec.: the time from O to A is 2 secs., what is the length of OA and how far is it beyond A after another 2 secs.?

9. The density per unit length at any point of a rod OA varies as the square of the distance from O. If the portion OP weighs W lbs. when $OP=x$ ins., it can be proved that $\frac{dW}{dx}=kx^2$ where k is a constant. If C is the mid-point of the rod and if the portion OC weighs 5 lbs., find the weight of CA.

10. Fig. 69 represents part of the graph of $y=(x-1)(4-x)$. If $ON=x$, $NP=y$ and A is the area bounded by AN, NP and the arc AP, it can be proved that

$$\frac{dA}{dx}=y=(x-1)(4-x).$$

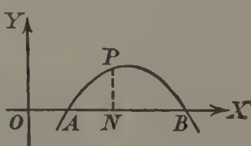

Fig. 69.

Find the area between the line AB and the arc AB.

11. A weight is suspended from a point O by a long spiral spring: its position of rest is A: it is pulled down 6 ins. and then let go. If its velocity is v ins. per sec. when it is x ins. from A, v and x are connected by the relation $\frac{d}{dx}(v^2) = -\frac{1}{2}x$. Find its velocity when passing A and the height above A to which it rises. How far should it be pulled down if it is to pass A with a velocity of 4 ins. per sec.?

12. A uniform chain OA 10 ins. long is being whirled round in a horizontal circle with centre O: the tension at a point P of the chain x ins. from O is T lbs. where $\frac{dT}{dx} = -kx$, k being a constant. The tension at O is 2 lbs., find k and the tension at the mid-point of the chain. [Note that the tension at A, the free end, must be zero.]

13. Prove that $y = 5x^3 - 7x$ is a solution of the equation $\frac{d^2y}{dx^2} = 30x$. What is the most general solution?

14. Express y in terms of x if (i) $\frac{d^2y}{dx^2} = x^2$, (ii) $\frac{d^2y}{dx^2} = \frac{1}{x^3}$.

15. Find y in terms of x if $\frac{d^2y}{dx^2} = 3x$ and if when $x = 1$, $\frac{dy}{dx} = 1$ and $y = 2$.

16. Given $\frac{dy}{dx} = y^2$, what is $\frac{dx}{dy}$? Express x in terms of y.

17. Express x in terms of y if $\frac{dy}{dx} = \sqrt{y}$ and $y = 1$ when $x = 1$.

18. A flexible uniform rod OA, 12 feet long, is supported at its ends which are at the same level. If the deflection NP is y when the distance ON is x, the relation between x and y is $\frac{d^2y}{dx^2} = 0 \cdot 0005\,(x^2 - 12x)$, the unit on each axis being 1 foot. Write down from symmetry the value of $\frac{dy}{dx}$ at the mid-point of the rod. Express y in terms of x. What is the slope of the rod at its ends? What is the maximum sag in inches?

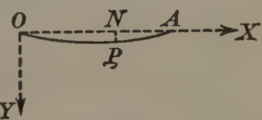

Fig. 70.

19. Water is being drawn up slowly at uniform speed from a well 16 feet deep in a bucket which leaks at a uniform rate. It starts with 20 lbs. of water in it and ends with 15 lbs. Find the work done in drawing up the bucket which if empty weighs 5 lbs. If E ft.-lbs. of work is done in drawing it up x feet, then $\frac{dE}{dx}$ equals the combined weight of the bucket and the water in it at that time.

20. A stone is thrown horizontally with a velocity of 30 feet a second from the top of a cliff 121 feet high. If after t seconds it has travelled x feet horizontally and y feet vertically, then neglecting air-resistance $\frac{d^2x}{dt^2}=0$ and $\frac{d^2y}{dt^2}=32$. Express x and y each in terms of t. Find how far it is from the foot of the cliff when it strikes the ground and how long it takes. In what direction is it moving after 2 seconds?

21. A heavy chain 6 feet long is held with a length of 2 feet hanging over the edge of the table and the rest coiled on the table near the edge. It is released and the coil runs out freely: if the velocity is v ft. per sec. when a length of x feet is hanging over the edge, it can be proved that $\frac{d}{dx}(v^2x^2)=64x^2$. Find the velocity when the whole chain is just clear of the table.

22. A particle of mass 4 ozs. is fastened to the mid-point P of an elastic string fixed at its ends to two points A, B on a horizontal table 4 feet apart; the tension of the string is 2 lbs. P is pulled a distance of 2 inches at right angles to AB and released. If its velocity is v ins. per sec. when at a distance of x inches from the line AB, it can be proved that $\frac{d}{dx}(v^2)+512x=0$. Find the velocity with which it passes through its equilibrium position.

23. If a gas is kept at constant pressure, the rate of decrease of density as the temperature increases within certain limits varies as the square of the density. If for temperatures 45° C. and 60° C. the densities are respectively 0·002 and 0·0016, express the density ρ in terms of the temperature θ° C.

24. A point starts from O and Ox is the tangent to its path at O. If the length s ins. of the arc it describes and the angle ψ° through which its direction of motion is turned are connected by the relation $\frac{d\psi}{ds}=\frac{s}{20}$, how far does it go before it is again moving parallel to Ox and in the same sense?

25. When a spring of natural length a ins. is stretched from a length x ins. to a length $x+\delta x$ ins. the work done is $\lambda(x-a).\delta x$ inch-lbs. approximately, where the error is small compared with δx and λ is a constant. Compare the work done in stretching it from length a to length $\frac{5a}{4}$ with that needed to stretch it from length $\frac{5a}{4}$ to length $\frac{3a}{2}$ ins.

CHAPTER VII

DEFINITE INTEGRALS

Example 1.

Given the graph of $y=3x^2+2x+4$, find the area bounded by the curve, the x-axis and the ordinates $x=a$ and $x=b$.

In the figure, $OE=a$, $OF=b$, EH and FK are the ordinates at E, F; it is required to find the area (A) bounded by the lines HE, EF, FK and the curve HK.

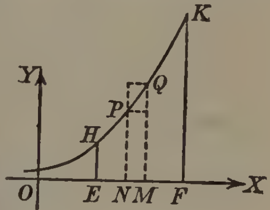

Fig. 71.

With the usual notation, let

$$ON=x, \quad NM=\delta x.$$

Imagine the ordinate PN to start at HE and to move parallel to itself with its ends on the curve and the x-axis, thus generating the area $HENP$ which we call z and which is a function of x. The object of the problem is to find this function.

δz represents the area between PN, NM, MQ and the arc PQ,

$$\therefore \ \delta z > NP \,.\, \delta x \ \text{ and } \ \delta z < MQ \,.\, \delta x,$$

$$\therefore \ \frac{\delta z}{\delta x} > y \ \text{ and } \ \frac{\delta z}{\delta x} < y + \delta y.$$

$\therefore$ in the limit when $\delta x \to 0$ we have

$$\frac{dz}{dx} = y = 3x^2 + 2x + 4,$$

$$\therefore \ z = \int (3x^2 + 2x + 4)\, dx$$

$$= x^3 + x^2 + 4x + c,$$

where c is an arbitrary constant.

The presence of this arbitrary constant is due to the fact that so far we have made no use of the fact that the area starts to be measured from where $x=a$.

This means that $z=0$ when $x=a$,

$$\therefore\ 0=a^3+a^2+4a+c,$$

$$\therefore\ c=-(a^3+a^2+4a),$$

$$\therefore\ z=(x^3+x^2+4x)-(a^3+a^2+4a).$$

When $x=b$, z will equal the area A which we require,

$$\therefore\ A=(b^3+b^2+4b)-(a^3+a^2+4a).$$

We will illustrate this principle by finding the area of the right-angled triangle OTR where

$$OT=TR=a.$$

Taking the X-axis along OT, we imagine the triangle to be generated by the motion of the ordinate of a point P which moves along the line OR.

At any instant, let the coordinates of P be (x, y) and let the area ONP then generated be z.

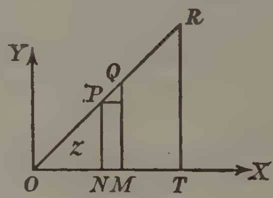

Fig. 72.

Now $\delta z=$ area of $NPQM$ and $\delta x=NM$.

But $\delta z > NP \,.\, \delta x$ and $< MQ \,.\, \delta x$.

$$\therefore\ \frac{\delta z}{\delta x} > y \text{ and } < y+\delta y.$$

$$\therefore\ \frac{dz}{dx}=y.$$

The equation of the line OR is clearly $y=x$,

$$\therefore\ \frac{dz}{dx}=x \text{ and } z=\int x\,.\,dx=\frac{x^2}{2}+c.$$

When $x=0$, z will be zero,

$$\therefore\ 0=0+c, \quad \therefore\ c=0,$$

$$\therefore\ z=\frac{x^2}{2}.$$

When $x=OT=a$, z will equal the area A which we require,

$$\therefore\ A=\frac{a^2}{2}.$$

EXAMPLES VII a

The argument of the previous example should be written out in full for each of the following:

Find the areas bounded by the X-axis, the given line or curve and the ordinates corresponding to the given abscissae:

1. $y=2x+3$ from $x=a$ to $x=b$;
2. $y=x^2$ from $x=0$ to $x=3$;
3. $y=5x^2+2$ from $x=-2$ to $x=1$;
4. $y=3x^2+2x+1$ from $x=-3$ to $x=-2$.

By dividing into strips parallel to OX, find the areas bounded by:

5. $y=2x+1$, the axis $x=0$, the lines $y=2$ and $y=5$;
6. $y=x^2$, the axis OY, the line $y=4$.

Notation

The symbol $\int_a^b f(x)\,dx$ is used to represent the value of $\int f(x)\,dx$ when $x=b$ *minus* the value of $\int f(x)\,dx$ when $x=a$.

Such an expression is called a *definite* integral, because the arbitrary constant automatically disappears in the subtraction.

Thus in the above example we see that

$$\text{area } HEFK = \int_a^b (3x^2+2x+4)\,dx.$$

Or again,

$$\int_1^2 (x^2-x)\,dx = \left[\frac{x^3}{3}-\frac{x^2}{2}+c\right]_1^2$$
$$= \left(\tfrac{8}{3}-\tfrac{4}{2}+c\right)-\left(\tfrac{1}{3}-\tfrac{1}{2}+c\right) = 2\tfrac{2}{3}-2-\tfrac{1}{3}+\tfrac{1}{2}=\tfrac{5}{6},$$

where we have used the notation $\Big[\phi(x)\Big]_a^b$ to represent $\phi(b)-\phi(a)$.

The symbol $\int$ used in integration is an elongated S which was originally written to denote that the area could be considered as the limit of the sum of an infinite number of rectangles, of which PM (Fig. 71) is a type.

The area $EHKF$ is given approximately by $\Sigma y\,\delta x$, which represents the sum of all the rectangles such as PM. The exact value of the area is given by the limit to which this series of rectangles continually approaches as δx is diminished and the number of rectangles correspondingly increased. By analogy with $\frac{dy}{dx}$, used as the limiting value of $\frac{\delta y}{\delta x}$, the limit of $\Sigma y\,\delta x$ is written $\int y\,dx$.

The method used in the above example is general, and leads to the following result:

The area bounded by the graph of $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$ is equal to $\int_a^b f(x)\,dx$ where $a<b$.

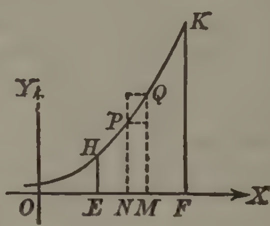

Fig. 73.

With the same notation as in the above example and by the same argument, we see that

$$\frac{\delta z}{\delta x} > y \text{ and } \frac{\delta z}{\delta x} < y + \delta y,$$

$\therefore$ in the limit when $\delta x \to 0$ we have $\frac{dz}{dx} = y = f(x)$,

$$\therefore\ z = \int f(x)\,dx + C,$$

where C is an arbitrary constant.

But $z=0$ when $x=a$,

$$\therefore\ C = -\text{the value of } \int f(x)\,dx \text{ when } x=a.$$

$$\therefore \text{ area of } HEFK = \text{the value of } \int f(x)\,dx \text{ when } x=b \textit{ minus}$$

$$\text{the value of } \int f(x)\,dx \text{ when } x=a$$

$$= \int_a^b f(x)\,dx.$$

Two points deserve notice:

(i) If $f(x)$ is a decreasing function of x, then

$$\delta z < y\,\delta x \text{ and } \delta z > (y+\delta y)\,\delta x,$$

or $$\frac{\delta z}{\delta x} < y \text{ and } > y+\delta y.$$

The inequalities are reversed, but the limiting result is the same, viz. $\dfrac{dz}{dx} = y$.

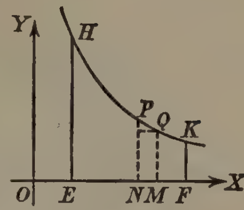

Fig. 74.

(ii) When part of the area is below the x-axis, the ordinates for that portion are negative. But δx is positive and δz lies between $y\,\delta x$ and $(y+\delta y)\,\delta x$,

$$\therefore\ \delta z \text{ is negative,}$$

$$\therefore \text{ the area below } Ox \text{ will be calculated as negative.}$$

Thus if in Fig. 75, $OE = a$, $OF = b$, the integral $\int_a^b f(x)\,dx$ will equal area EGH − area FGK.

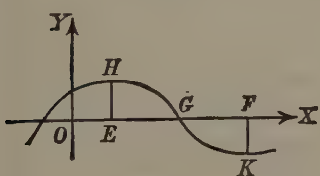

Fig. 75.

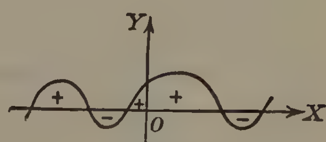

Fig. 76.

Areas to the *left* of Oy and *above* Ox (see Fig. 76) are of course calculated as positive because both y and δx are positive: but areas to the *left* of Oy and *below* Ox are negative, since y is negative.

Note. The condition $a < b$ is equivalent to taking δx positive. This is the usual convention because it makes the sign of the area, as represented by the integral, the same as the sign of y.

EXAMPLES VII b

AREAS

1. Find the values of

(i) $\int_0^2 x\,dx$; (ii) $\int_1^2 x^2 dx$; (iii) $\int_1^3 (x^3 - x^2)\,dx$;

(iv) $\int_1^2 \frac{dx}{x^2}$; (v) $\int_{-1}^1 (1+x)^2 dx$; (vi) $\int_1^4 \sqrt{x}\,dx$;

(vii) $\int_{-2}^3 (2x^2-7)\,dx$; (viii) $\int_0^8 \sqrt[3]{x}\,dx$.

2. Evaluate:

(i) $\int_{-h}^h (a+bx+cx^2)\,dx$; (ii) $\int_{-1}^1 x^2(1-x)\,dx$;

(iii) $\int_1^2 \pi y^2 dx$ if $xy=c^2$; (iv) $\int_{10}^{20} p\,.\,dv$ where $p\,.\,v^{1\cdot 4}=5$.

3. Fig. 77 represents the graph of $y=x^2$. PN, PL are the perpendiculars from P to Ox, Oy. If $ON=2$, find the area bounded by

(i) ON, NP, arc OP; (ii) OL, LP, arc OP.

Evaluate $\int_0^4 x\,dy$.

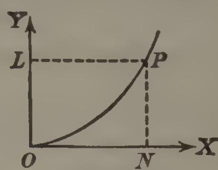

Fig. 77.

4. Find the area bounded by the x-axis, the ordinates $x=1$, $x=4$ and the curve $y^2=x$.

5. Draw the graph of $y=3x$ and interpret geometrically $\int_1^2 3x\,.\,dx$. Evaluate this integral in two distinct ways.

6. Fig. 78 represents the graph of $y=4x-x^2$: (i) Find the area between Ox and the part of the curve above Ox; (ii) Evaluate $\int_0^6 (4x-x^2)\,dx$ and explain the result.

Fig. 78.

7. If Fig. 77 represents the graph of $y=x^3$ and if $ON=a$, find (i) the area bounded by ON, NP, arc OP; (ii) the ratio in which the curve OP divides the area of the rectangle $ONPL$.

8. Find the area bounded by the x-axis and the portion of the curve $y=(x-1)(x-3)$ which lies below it.

9. Find the area bounded by the curve $y=\frac{1}{x^2}$, the y-axis and the lines $y=1$, $y=4$.

10. Find the area bounded by the curve $y=x^2+4x+3$, the lines $x=1$, $x=2$ and $y=1$.

11. Find the areas of the segments cut off by the x-axis from the curves

(i) $y=x(x-1)(x-2)$; (ii) $y=x(x-1)(x+3)$.

12. Find the area bounded by the x-axis, the ordinates $x=1$ and $x=4$ and the curve $y=x^2+\frac{1}{x^2}$.

13. A curve passes through the origin and its gradient at any point (x, y) is $1-\frac{1}{3}x^2$. Find the area bounded by the curve, the x-axis and the ordinates $x=1$, $x=2$.

14. Fig. 79 represents the graph of $y=x^2(3-x)$; a line is drawn parallel to Ox touching the curve at P and cutting it again at Q. Calculate (i) the coordinates of P, Q; (ii) the area of segment OPA; (iii) the area of segment POQ.

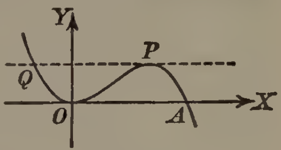

Fig. 79.

15. Find the area bounded by the x-axis and the portion of the graph of $y=x^2(x-2)(x-5)$ which lies below it.

16. Prove that the line $y=2x$ cuts the parabola $3y=x^2$ at the point $(6, 12)$ and find the area of the segment the line cuts off from the curve.

17. Find the area of the segment cut off from the curve $y=x(2-x)$ by the line $2y=x$.

18. Show that the curves $y^2=x$ and $x^2=8y$ cut at the point $(4, 2)$ and find the area intercepted between them.

19. A trough 10 feet long has a parabolic section: it is 20 inches deep and 30 inches wide at the top. Taking the equation of the parabola as $x^2=Cy$, find the value of C and the volume of the trough in cubic feet. [Unit of each axis, 1 foot.]

20. Evaluate and interpret geometrically:

(i) $\int_{-2}^{-1}(x+1)(3-x)\,dx$; (ii) $\int_{-1}^{0}(x+1)(3-x)\,dx$;

(iii) $\int_{0}^{3}(x+1)(3-x)\,dx$; (iv) $\int_{3}^{4}(x+1)(3-x)\,dx$.

21. Fig. 80 represents the graph of $y^2=x(x-4)^2$. Find the area of the loop OA.

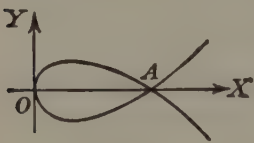

Fig. 80.

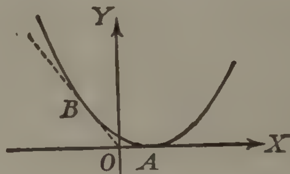

Fig. 81.

22. Find the area bounded by the curve $y^2=x^3$ and the line $x=4$.

23. The graph of $y=(x-1)^2$ (see Fig. 81) touches Ox at A. Show that it touches $y=-4x$ at the point $B\ (-1, 4)$: and find the area enclosed by the curve and the tangents OA, OB.

24. Find the area between the curves $x^2y=4$, $x^3y=8$ and the line $y=2$.

25. Illustrate geometrically the identities:

(i) $\int_a^b f(x)\,.\,dx=\int_a^c f(x)\,.\,dx+\int_c^b f(x)\,.\,dx$;

(ii) $\int_a^b f(x)\,.\,dx=\int_{a-c}^{b-c} f(x+c)\,.\,dx$.

26. Show geometrically that if $a<b$ and if M and m are the greatest and least values of $f(x)$ as x varies from a to b, then

$$M(b-a)>\int_a^b f(x)\,.\,dx>m(b-a).$$

Volumes of Solids of Revolution

When a curve rotates about an axis, it generates the surface of a solid of revolution: in such cases the section of the solid at right angles to the axis is a circle and the volume can be obtained as an integral.

Example 2.

Let (x, y) be the coordinates of any point P on the curve $y=f(x)$ which is rotated about Ox to form a solid of revolution. It is required to find the volume of the solid between two planes perpendicular to Ox which cut Ox at A, B where $OA=a$, $OB=b$. Let V be the volume of the solid from the plane through A up to the plane through P perpendicular to Ox. Let Q be $(x+\delta x, y+\delta y)$: then δV is the volume of the solid between the planes through P and Q. The face through P is a circle of radius $PN=y$ and of area πy^2; the face through Q is similarly a circle of radius $y+\delta y$ and area $\pi(y+\delta y)^2$. The distance between the two faces is δx.

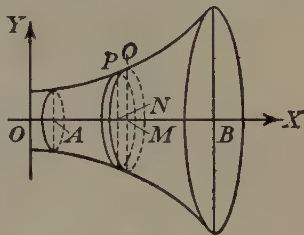

Fig. 82.

$$\therefore\ \delta V > \pi y^2 . \delta x \text{ and } \delta V < \pi(y+\delta y)^2 . \delta x,$$

$$\therefore\ \frac{\delta V}{\delta x} \text{ lies between } \pi y^2 \text{ and } \pi(y+\delta y)^2,$$

$\therefore$ in the limit when $\delta x \to 0$, since δy also $\to 0$, we have

$$\frac{dV}{dx}=\pi y^2.$$

$$\therefore\ V=\int \pi y^2 dx.$$

Since $V=0$ when $x=a$, the volume between the sections through A and N is $\int_a^x \pi y^2 dx$.

$\therefore$ the volume between the sections through A and B is $\int_a^b \pi y^2 dx$ where $y=f(x)$.

Example 3.

Find the volume of a spherical segment of height h cut from a sphere of radius r.

In Fig. 83 the rotation of arc PB about Ox forms a spherical segment of height NB,

$$\therefore\ ON=OB-NB=r-h.$$

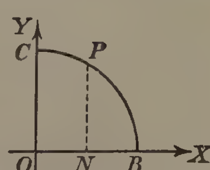

Fig. 83.

$$\therefore \text{ volume} = \int_{r-h}^{r} \pi y^2 dx = \pi \int_{r-h}^{r} (r^2 - x^2)\, dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^{r} = \pi \left[r^3 - \frac{r^3}{3} \right] - \pi \left[r^2 (r-h) - \frac{(r-h)^3}{3} \right]$$

$$= \pi \left\{ \frac{2r^3}{3} - r^3 + r^2 h + \frac{r^3 - 3r^2 h + 3rh^2 - h^3}{3} \right\}$$

$$= \frac{\pi}{3} \{3rh^2 - h^3\} = \frac{\pi h^2 (3r-h)}{3}.$$

If $h = r$ we have the volume of a hemisphere $= \dfrac{2\pi r^3}{3}$ and $\therefore$ the volume of a sphere is $\frac{4}{3}\pi r^3$.

It is sometimes convenient to slice up the solid in a different way: we shall illustrate this method by finding the volume of a circular cone.

Example 4.

To find the volume of a circular cone radius a, height h ins.

A is the vertex of the cone, O is the centre of its base BC: $OA = h$, $OC = a$. Divide the base into a series of concentric rings. Let the radii ON, OM of the inner and outer edges of one of these rings be r, $r + \delta r$ ins.

We regard the cone as sliced up into a number of thin cylindrical collars, the base of each being one of these concentric rings. The area of the base of the ring in the figure is approximately $2\pi r \,.\, \delta r$.

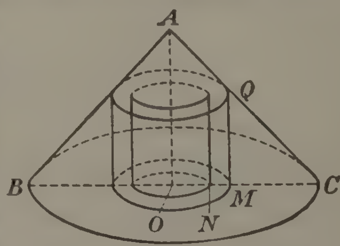

Fig. 84.

Also $$\frac{QM}{MC} = \frac{AO}{OC} = \frac{h}{a}, \quad \therefore\ QM = \frac{h}{a}(a - r - \delta r),$$

$\therefore$ the volume of this collar is approximately

$$2\pi r \,.\, \delta r \,.\, \frac{h}{a}(a - r) \text{ cu. ins.}$$

But the volume of the cone is obtained by taking the limit of the sum of these collars as their number increases indefinitely.

$$\therefore \text{ the volume} = \int_0^a \frac{2\pi h}{a} r(a-r)\,dr$$
$$= \frac{2\pi h}{a}\int_0^a (ar - r^2)\,dr$$
$$= \frac{2\pi h}{a}\cdot\left[\frac{ar^2}{2} - \frac{r^3}{3}\right]_0^a = \frac{2\pi h}{a}\left(\frac{a^3}{2} - \frac{a^3}{3}\right)$$
$$= \frac{\pi h a^2}{3} = \tfrac{1}{3}\pi a^2 h.$$

EXAMPLES VII c

Volumes

1. The line $y=2x$ is rotated about the x-axis to form a circular cone: find the volume of the portion from $x=0$ to $x=5$.

2. The arc of the parabola $y^2=4x$ between $(0, 0)$ and $(1, 2)$ is rotated (i) about the x-axis, (ii) about the y-axis. Find the volumes of the solid of revolution generated in each case.

3. Sketch the graph of $xy=12$: the portion of the curve from $x=2$ to $x=3$ is rotated about the x-axis: find the volume of the solid so generated.

4. A basin is formed by the revolution of $y=\frac{1}{50}x^4$ about the y-axis (unit on each axis 1″): if the radius of the rim is 5″, find the depth and its cubic content.

5. The loop of $y^2=x(x-4)^2$ (see Fig. 80) is rotated about the axis of x; what is the volume of the solid so formed?

6. The area cut off from the hyperbola $x^2-y^2=a^2$ by the chord $x=b$ is revolved round the axis OY. Find the volume of the solid thus generated.

7. A beer barrel is formed by the rotation of the part of the curve $y=\frac{1}{10}(x+2)(8-x)$ between $x=0$ and $x=6$ about the x-axis: unit on each axis $\frac{1}{2}$ foot. Find the volume of the barrel in cu. ft.

8. A cylindrical block 4″ long is turned in a lathe till its shape is that of the surface of revolution formed by rotating the part of the curve $y=\frac{1}{4}x^2+2$ between $x=2$ and $x=-2$ about the x-axis, unit on each axis 1″. Find the volume of the finished block.

9. The head of a bullet is formed by the revolution of part of the parabola $y = Cx^2$ about the y-axis, starting from the origin. If the total length of head is 2·7 cms. and the diameter of the bullet is 0·6 cm., find C and the volume of the head of the bullet. [Unit on each axis, 1 cm.]

10. A vertical column 20 feet high is of square horizontal section: at a height of x feet from the ground the side of the square is

$$10\left(1 - \frac{x^2}{400}\right) \text{ feet.}$$

Find the volume of the column.

11. ABC is a triangle such that $AB = BC = 10''$, $\angle ABC = 90°$: EF is a line parallel to AB and at distances $16''$ and $26''$ from B, C respectively. What is the volume of the solid obtained by rotating the triangle ABC about the line EF?

12. The base of a solid is a circle, centre O, radius a ins.; if N is any point on the base at a distance r ins. from O, the height of the solid at N is $\frac{1}{a}(a^2 - r^2)$ ins. Find the volume of the solid. [Use the method of Ex. 4, p. 92.]

13. A spherical bowl of radius 2 feet contains water to a depth of $6''$; how much water must be poured in to increase the depth by $6''$?

14. The base of a solid is a quadrant AB of a circle centre O, radius $4''$: if P is any point on the base, the height of the solid above P is $\frac{1}{4}\sqrt{OP}$ inches. Find the volume of the solid.

15. The base of a mound is a rectangle $15'$ by $10'$ and its top is a rectangle in a parallel plane $5'$ square; its sides are plane and its height is $12'$; prove that the area of a plane section x feet above the base is $\frac{25}{72}(18 - x)(24 - x)$ sq. ft. and find the volume of the mound.

16. Two semi-circles are described on a line AB $6''$ long in planes at right angles: a solid is formed by joining points on the one semi-circle to the corresponding points on the other by straight lines. Find the volume of the solid.

17. A rod AB of variable density, 5 feet long, is such that the material at a point P, x feet from A, weighs $1 + 3x$ lbs. per foot length of rod. If the portion AP weighs W lbs., prove that $\frac{dW}{dx} = 1 + 3x$ and find the weight of the rod.

18. The density of a solid spherical body varies as its distance from the centre. When the radius is 2 feet, the body weighs 10 lbs. What would be the least weight if the radius is reduced to 1 foot?

19. The axes of two circular cylinders each of radius a ins. cut at right angles. Fig. 85 represents the section through their axes. Show that the section common to both cylinders made by a plane parallel to and at distance x ins. from this section is a square of area $4(a^2-x^2)$ sq. ins. and find the total volume common to both cylinders. (Trin. Coll.)

Fig. 85.

CHAPTER VIII

APPROXIMATE EVALUATION OF DEFINITE INTEGRALS

We have seen that any definite integral $\int_a^b f(x) \, . \, dx$ can always be regarded as representing the area between the graph of $y = f(x)$, the x-axis and the ordinates

$$x = a, \; x = b.$$

If therefore we wish to find the approximate value of some definite integral it is only necessary to consider methods for the approximate evaluation of the corresponding area.

Fig. 86.

The Trapezoidal Rule

Consider the area bounded by the curve, the ordinates AH, BK and the line AB.

Divide AB into any number n equal parts and let the corresponding ordinates at A and the points of section be y_1, y_2, y_3, ... Let $\dfrac{AB}{n} = h$. Join the upper ends of the ordinates. We shall take the sum of the areas of the trapezia so obtained as an approximation for the required area.

This gives

$$\tfrac{1}{2} h (y_1 + y_2) + \tfrac{1}{2} h (y_2 + y_3) + \tfrac{1}{2} h (y_3 + y_4) + \ldots + \tfrac{1}{2} h (y_n + y_{n+1})$$
$$= h \left[\tfrac{1}{2} (y_1 + y_{n+1}) + (y_2 + y_3 + \ldots + y_n) \right].$$

We may state this approximate rule as follows:

Divide up the area by any number of equidistant ordinates. Take the average of the first and last ordinate and add to it the sum of the other ordinates, multiply the result by the distance between two consecutive ordinates.

Example 1.

Use the trapezoidal rule to evaluate

$$\int_2^8 x^2(10-x)\,dx.$$

Fig. 87 represents the graph of

$$y=x^2(10-x)=10x^2-x^3.$$

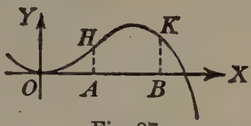

Fig. 87.

We have the following table of values:

$x=2$	3	4	5	6	7	8
$10x^2-x^3=32$	63	96	125	144	147	128

$\therefore$ the trapezoidal rule gives for the area

$$(1)\,[\tfrac{1}{2}(32+128)+63+96+125+144+147]$$
$$=\tfrac{1}{2}(160)+575=80+575=655.$$

By Calculation

$$\text{integral}=\left[\frac{10x^3}{3}-\frac{x^4}{4}\right]_2^8=\frac{5120-80}{3}-1020=660.$$

$\therefore$ the error is less than 1 per cent.

It is obvious that an increase in the number of ordinates decreases the error.

Simpson's Rule

Suppose PQR is an arc of the curve

$$y=a+bx+cx^2$$

which is a parabola with its axis parallel to OY and suppose the Y-axis is taken through Q and that the ordinates PL, QO, RM are equidistant, $LO=OM=h$.

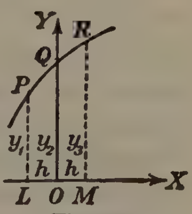

Fig. 88.

Then, if $LP=y_1$, $OQ=y_2$, $MR=y_3$, we shall prove that the area

$$LMRQP=\tfrac{1}{3}h(y_1+y_3+4y_2).$$

$$\text{The area}=\int_{-h}^{h}(a+bx+cx^2)\,dx=\left[ax+\tfrac{1}{2}bx^2+\tfrac{1}{3}cx^3\right]_{-h}^{h}$$
$$=(ah+\tfrac{1}{2}bh^2+\tfrac{1}{3}ch^3)-(-ah+\tfrac{1}{2}bh^2-\tfrac{1}{3}ch^3)$$
$$=2ah+\tfrac{2}{3}ch^3.$$

But the points

$$(-h, y_1),\ (0, y_2),\ (h, y_3) \text{ lie on } y = a + bx + cx^2.$$

$$\therefore\ y_1 = a - bh + ch^2;\ y_2 = a;\ y_3 = a + bh + ch^2.$$

$$\therefore\ y_1 + y_3 = 2a + 2ch^2 = 2y_2 + 2ch^2.$$

$$\therefore\ 2ch^2 = y_1 + y_3 - 2y_2.$$

$$\begin{aligned}\therefore \text{ the area } &= 2y_2 h + \tfrac{1}{3} h (y_1 + y_3 - 2y_2)\\ &= \tfrac{1}{3} h (6y_2 + y_1 + y_3 - 2y_2)\\ &= \tfrac{1}{3} h (y_1 + y_3 + 4y_2).\end{aligned}$$

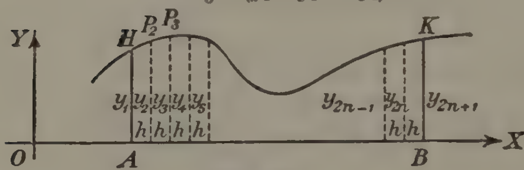

Fig. 89.

Consider now the area $ABKH$; divide AB into any *even* number of parts, say $2n$; and let the ordinates through A, the points of section, and B be $y_1,\ y_2,\ y_3, \dots y_{2n+1}$, and their extremities $H,\ P_2,\ P_3, \dots P_{2n},\ K$. Replace the arc HP_2P_3 by a parabolic arc through $H,\ P_2,\ P_3$ with axis parallel to OY; similarly replace the arc $P_3P_4P_5$ by another parabolic arc through $P_3,\ P_4,\ P_5$ and so on. Let $\dfrac{AB}{2n} = h$. The area enclosed by this curve and HA, AB, BK is

$$\begin{aligned}&\tfrac{1}{3} h (y_1 + y_3 + 4y_2) + \tfrac{1}{3} h (y_3 + y_5 + 4y_4) + \dots + \tfrac{1}{3} h (y_{2n-1} + y_{2n+1} + 4y_{2n})\\ &= \tfrac{1}{3} h \left[y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + \dots + 2y_{2n-1} + 4y_{2n} + y_{2n+1}\right]\\ &= \tfrac{1}{3} h \left[y_1 + y_{2n+1} + 2(y_3 + y_5 + y_7 + \dots + y_{2n-1}) + 4(y_2 + y_4 + \dots + y_{2n})\right].\end{aligned}$$

This is Simpson's approximate expression for the area of a curve, and it can be written down by using the following rule:

Divide the area into any *even* number of strips of equal breadth whose sides are parallel to the y-axis:

To the sum of the first and last ordinate, add twice the sum of the remaining odd ordinates and four times the sum of all the even ordinates; multiply the result by one-third of the breadth of any strip.

Various other approximate methods have been suggested. Two of these are shown below:

Dufton's modifications of Gauss' Rule

(i) Fig. 90. Divide AB into ten equal parts

$$AA_2,\ A_2A_3,\ \ldots A_{10}B,$$

and draw the ordinates at A_2, A_5, A_7, A_{10}. Then the area is approximately equal to $\frac{1}{4}AB(y_2 + y_5 + y_7 + y_{10})$.

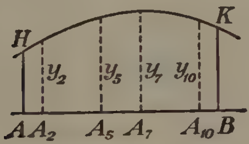

Fig. 90.

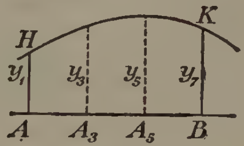

Fig. 91.

(ii) Fig. 91. Divide AB into six equal parts

$$AA_2,\ A_2A_3,\ \ldots A_6B,$$

and draw the ordinates at A_3, A_5.

Then the area is approximately equal to

$$\tfrac{1}{8}AB(y_1 + 3y_3 + 3y_5 + y_7).$$

Example 2.

Find by using (i) Dufton's two rules, (ii) Simpson's rule the approximate value of $\int_1^2 \frac{dx}{x}$.

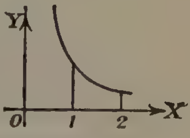

Fig. 92.

By Dufton's rule (i):

the integral $= \frac{1}{4}(\cdot 9091 + \cdot 7143 + \cdot 6250 + \cdot 5263) = 0\cdot 6936$.

By Dufton's rule (ii):

$$\text{the integral} = \tfrac{1}{8}\left(1 + \frac{3}{1\frac{1}{3}} + \frac{3}{1\frac{2}{3}} + \tfrac{1}{2}\right) = \tfrac{1}{8}\left(1 + \tfrac{9}{4} + \tfrac{9}{5} + \tfrac{1}{2}\right)$$

$$= \tfrac{1}{8}(1 + 2\cdot 25 + 1\cdot 8 + 0\cdot 5) = 0\cdot 6937.$$

With the notation used above for Simpson's rule we have

x	values of $\frac{1}{x}$		
		even ordinates	other odd ordinates
1	1		
1·1		·9091	
1·2			·8333
1·3		·7692	
1·4			·7143
1·5		·6667	
1·6			·6250
1·7		·5882	
1·8			·5556
1·9		·5263	
2·0	·5		
	1·5	3·4595	2·7282
Multipliers		4	2
		13·8380	5·4564

1·5
13·838
5·456
20·794

$\therefore \int_1^2 \frac{dx}{x}$ equals approximately $\frac{1}{3} \times 0{\cdot}1 \times 20{\cdot}79$ or $0{\cdot}693$.

(The correct value to 6 figures is 0·693147.)

EXAMPLES VIII a

1. Find approximately the area $ABKH$ in Fig. 93 using

(i) the trapezoidal rule (10 strips),

(ii) Simpson's rule (10 strips),

(iii) Dufton's first rule.

2. Write down the values of x^2 for $x=0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3, \ldots\ldots 0{\cdot}9$, $1{\cdot}0$ and find by *three* approximate methods the area between the curve $y=x^2$, the x-axis and the ordinates $x=0$, $x=1$. Compare the results with the value of $\int_0^1 x^2\,dx$.

3. Draw a semi-circle with diameter 10 cms. and find by measurement, using (i) Dufton's second rule, (ii) Simpson's rule (8 strips), its area and estimate the approximate error per cent. in each case.

4. Find an approximate value of $\int_0^6 y\,dx$, given that

$x=0$	1	2	3	4	5	6
$y=3$	7	15	13	10	6	1

5. Find an approximate value of $\int_1^2 \sqrt{1+x^2}\,dx$.

6. Find an approximate value of $\int_0^1 10^x\,dx$.

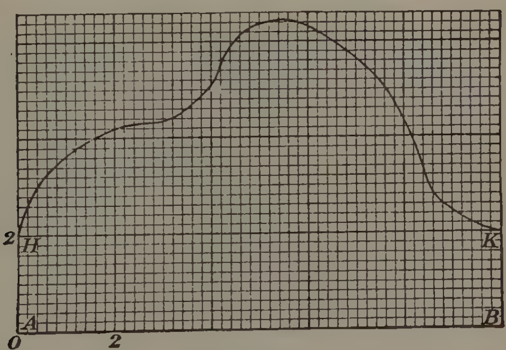

Fig. 93.

Further Applications of Simpson's Rule

The simplest use of this rule is its application to the evaluation of areas, but it may equally well be employed in any other problem which involves the computation of a definite integral.

Example 3.

The areas of the cross-sections of a body at right angles to its axis OA are shown in the following table:

Distance of section from O in ins.	$=x$	0	1	2	4·2	5·4	6·4	7	8	9·2	9·8	11	12
Area of section in sq. ins.	$=A$	0	18	32	54	62	66	68	70	68	60	38	0

Find approximately the volume of the body.

If A was a given function of x, the volume would be obtained by evaluating the definite integral $\int_0^{12} A \, . \, dx$.

In order to apply Simpson's rule to this integral we must divide the interval $0 \longleftrightarrow 12$ into an even number of strips and obtain the corresponding values of A at the points of section of the interval.

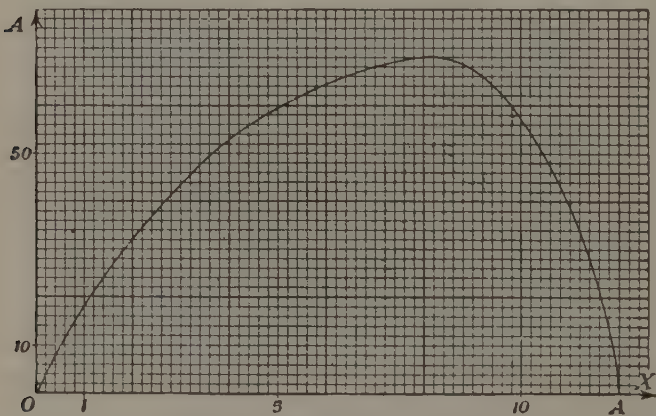

Fig. 94.

Fig. 94 represents the graph obtained by plotting the values in the given table. From the graph we can read off the required values of A; these are as follows:

$x=0$	1	2	3	4	5	6	7	8	9	10	11	12
$A=0$	18	32	43·5	52·5	59·5	64	68	70	67·5	57	38	0

$\therefore$ by Simpson's rule, $\int_0^{12} A\,.\,dx$

$=\frac{1}{3}[0+2(32+52·5+64+70+57)+4(18+43·5+59·5+68+67·5+38)]$
$=\frac{1}{3}[2\times 275·5+4\times 294·5]=\frac{1}{3}[551+1178]=\frac{1}{3}\times 1729$
$=576$ cu. ins.

We will compare this result with that obtained by applying Dufton's second rule to the intervals $0 \leftrightarrow 6$ and $6 \leftrightarrow 12$ successively and adding the two results. We then have

$$\tfrac{1}{8}\times 6[0+3(32+52·5)+64]+\tfrac{1}{8}\times 6[64+3(70+57)+0]$$
$$=\tfrac{3}{4}[3(84·5+127)+128]=\tfrac{3}{4}(634·5+128)=\tfrac{3}{4}\times 762·5$$
$$=572 \text{ cu. ins.}$$

Note. (1) Both for the sake of speed and accuracy it is advisable to arrange the working on the plan shown in Example 2, p. 100, and not as in Example 3 above.

(2) Simpson's rule being the evaluation of $\int(a+bx+cx^2)\,dx$ it follows that when used for the evaluation of volumes, if A is of the form $a+bx+cx^2$ then Simpson's rule will give an exact value and not an approximation. In such cases we need only take three ordinates.

EXAMPLES VIII b

1. A curve passes through the following points:

$x=0$	6	15	21	26	30	35	40
$y=10$	14	9	5	5	7	7	0

Find the area between the curve and the axes of reference.

2. Find the volume of a solid whose axis is 10 ins. long, the areas of the cross-sections perpendicular to the axis being as follows:

Distance along axis in inches	$=x=0$	2	4	6	8	10
Area of cross-section in sq. inches	$=A=25$	32	40	50	35	25

3. If a variable force acts on a body of mass m lbs. initially at rest for t seconds, the body acquires a velocity of v feet per second where $\frac{mv}{32}=\int_0^t P\,dt$ where P lbs. is the force after t seconds. Find the velocity acquired in 12 secs. by a body of mass 20 lbs. from the following data:

$t=0$	2	4	6	8	10	12
$P=5$	9	16	25	20	12	0

4. The velocity of a train, which starts from rest, at time t minutes after the start is v miles an hour where v, t are given in the following table. How far does it travel in 16 minutes?

$t=0$	2	4	6	8	10	12	14	16
$v=0$	12	26	32	35	36	25	10	0

5. If the cross-section of a solid at distance x ins. along its axis is S sq. ins. where $S=ax^2+bx+c$, and if S_1, S_2, S_3 are the values of S when $x=-h$, 0, h respectively, show that the volume between $-h$ and h is $\frac{1}{3}h(S_1+S_3+4S_2)$.

6. Use Simpson's rule to prove that the volume of a cone is $\frac{1}{3}\pi r^2h$.

7. Use Simpson's rule to prove that the volume of a sphere is $\frac{4}{3}\pi r^3$.

8. The figure is a sketch of a railway embankment, the top of which $XYZW$ is level. The strips shown are of equal width, 20 feet.

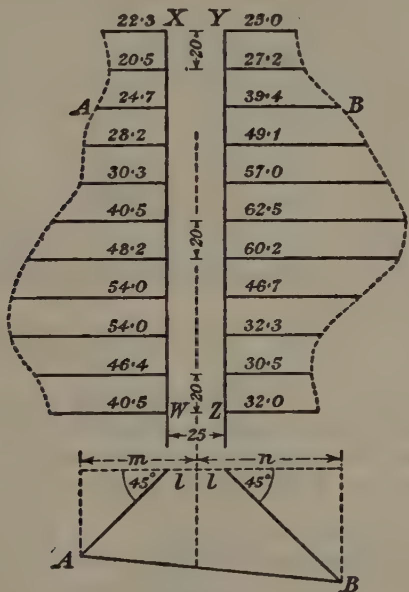

Fig. 95.

The dimensions given in the figure represent horizontal distances (in feet) measured from XW and YZ on opposite sides.

At any cross-section AB, the original surface of the ground is straight but sloping and the sides of the embankment slope at 45°. Show that, $2l$ being the breadth at the top, and m and n the horizontal distances of A and B from the centre line, the area of the section is $mn - l^2$. Hence by Simpson's rule find the volume of the embankment. (Army 1909.)

Mean values

We shall now consider what meaning (if any) can be attached to the average value of $f(x)$ as x varies from a to b.

Let PQ be the graph of $f(x)$ from $x=a$ to $x=b$. Divide the interval $b-a$ into n equal parts and let each part be h; and suppose the corresponding ordinates to be $y_1, y_2, y_3, \dots y_{n+1}$.

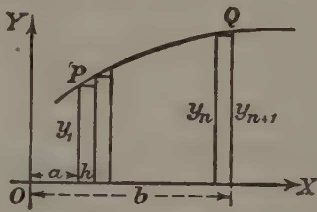

Fig. 96.

The arithmetic average of these ordinates will be

$$\frac{y_1+y_2+y_3+\dots+y_{n+1}}{n+1}=\frac{(y_1+y_2+y_3+\dots+y_{n+1})h}{nh+h}$$
$$=\frac{(hy_1+hy_2+\dots+hy_n)+h\,.\,y_{n+1}}{(b-a)+h}.$$

The numerator of this fraction consists of the sum of the inscribed rectangles $+h(y_{n+1})$ and as $h\to 0$ the limit of this expression will be the area under the graph between $x=a$ and $x=b$, i.e. the limit of the average of ordinates taken *equally spaced* when the number of ordinates increases without limit is $\dfrac{\int_a^b f(x)\,dx}{b-a}$.

This is called the *mean value* of the function $f(x)$ in the interval $a \leftrightarrow b$ taken *with respect to x*.

The geometrical interpretation of this result is both simple and important.

Draw a line RS parallel to OX cutting the ordinates AH, BK at R, S so that the area of the rectangle $RABS$ equals the area of the curve $HABK$.

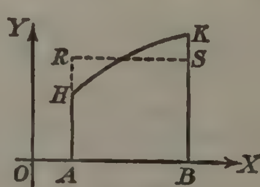

Fig. 97.

Then $AR\,.\,AB=\int_a^b f(x)\,.\,dx$,

or $AR=\dfrac{1}{b-a}\int_a^b f(x)\,.\,dx=$ mean value of $f(x)$ with respect to x.

The mean value of a function depends on the variable with respect to which it is calculated. Suppose for example a stone falls from the top of a tower 144 feet high. If we neglect the air pressure and take the acceleration as 32 feet per sec. per sec., we have with the usual notation

$$\frac{dv}{dt} = 32 \text{ and } \therefore\ v = 32t \text{ since } v = 0 \text{ when } t = 0,$$

also $$\frac{ds}{dt} = v = 32t, \quad \therefore\ s = 16t^2 \text{ since } s = 0 \text{ when } t = 0.$$

But when $t = 3$, $s = 16 \times 9 = 144$, so that it reaches the ground in 3 seconds.

What do we mean by the average velocity of the stone during its fall?

(i) Suppose we calculate its velocity at *equal intervals of time* elapsed since the start.

$$\begin{aligned} \text{Then its average velocity} &= \tfrac{1}{3}\int_0^3 v\,dt \\ &= \tfrac{1}{3}\int_0^3 32t\,.\,dt \\ &= \tfrac{1}{3} \times \Big[16t^2\Big]_0^3 = 48 \text{ ft. per sec.} \end{aligned}$$

(ii) Suppose we calculate its velocity at *equal intervals of distance* fallen since the start.

Then its average velocity $= \frac{1}{144}\int_0^{144} v\,ds$.

But $v^2 = (32t)^2 = 64 \times 16t^2 = 64s$ or $v = 8 \times s^{\frac{1}{2}}$.

$$\begin{aligned} \therefore \text{ average velocity} &= \tfrac{1}{144}\int_0^{144} 8 \times s^{\frac{1}{2}}\,ds \\ &= \tfrac{1}{144} \times 8 \times \tfrac{2}{3}\Big[s^{\frac{3}{2}}\Big]_0^{144} \\ &= \frac{16}{144 \times 3} \times 144^{\frac{3}{2}} = \frac{16}{144 \times 3} \times 144 \times 12 \\ &= 64 \text{ ft. per sec.} \end{aligned}$$

Therefore its mean speed with respect to the time is 48 ft. per sec. and its mean speed with respect to the distance is 64 ft. per sec.

It is not surprising the two results are different because the velocity increases in this special case *uniformly* with the time but *not uniformly* with the distance.

EXAMPLES VIII c

1. Find the mean value of x^3 in the interval $x=1$ to $x=2$ for uniform increases in x.

2. A circular cone has base-diameter 6″ and height 3″. Sections are formed by a series of planes parallel to the base and equally spaced. What is the diameter of the mean section?

3. The radius of a spherical envelope is expanding from 1″ to 2″. (i) Find the area of the mean surface for equal increments of the radius; (ii) Find the area of the mean surface for equal increments of volume.

4. The density at any point of a solid sphere at distance r from the centre is $\frac{kr}{a}$, where a is the radius of the sphere and k is a constant. What is the mean density of the sphere?

5. If $v=u+pt+qt^2$ is the relation between the speed v of the body and the time t for which it has been moving, find the mean value of v between the times t_1 and t_2 and compare the result with the mean value of the speed at t_1 and the speed at t_2.

6. If a body acquires a velocity v feet per sec. after falling t secs. from rest and if the distance fallen in this time is s feet, then $v=32t$ and $s=16t^2$. Find the mean speed after t secs. (i) for equal intervals of time, (ii) for equal intervals of distance.

7. A spring of natural length l ins. is being stretched slowly. When the stretched length is $l+x$ ins., the tension at the end of the spring is $\frac{kx}{l}$ lbs. where k is a constant. What is the mean tension for equal intervals of distance when the spring is stretched from l to $l+a$ ins.?

8. A gas expands according to the law $p \,.\, v^{1\cdot 4}=k$. If $p=200$ lbs. per sq. ft. when $v=2$ cu. ft., find the mean value of the pressure between $v=2$ and $v=5$ for equal increments of v.

REVISION PAPERS 6—11

R. 6

1. Integrate with respect to x:

$$\text{(i) } 3x^3-\tfrac{1}{2}x; \quad \text{(ii) } \frac{3x-2}{x^3}; \quad \text{(iii) } \frac{1}{x^{0\cdot 1}}; \quad \text{(iv) } (2-5x)^3.$$

2. Find the equation of a curve which passes through the origin and whose gradient is $2x^2-3x+1$.

3. Write down the equation of the tangent and the normal to $y=x(3-x)$ at the point (1, 2) and find the lengths of the subtangent and the subnormal.

4. If $v=8+32t-t^2$ gives the velocity v in terms of the time t, find the distance s travelled when $t=30$ given that $s=22$ when $t=0$.

5. Find the value of x for which the functions

$$x^3-3x \quad \text{and} \quad x^3-5x^2+17x$$

change at the same rate.

R. 7

1. When $x=6$ the abscissa of the parabola $6y=x^2$ is changing at 2 ft./sec.; find the rate at which the ordinate is then changing. [Unit on each axis, 1 foot.]

2. Sketch a curve whose equation is $y=f(x)$ such that for positive values of x, $f(x)$ decreases as x increases but the rate of decrease of $f(x)$ diminishes as x increases.

3. Find from first principles, without using the sign $\int$, the area between the curve $y=x^2-2x$ and OX.

4. Evaluate: (i) $\int_2^3 \frac{dx}{x^2}$; (ii) $\int_1^9 \sqrt{x}\,dx$; (iii) $\int_{-1}^{+1}(1+x)^3\,dx$;

(iv) $\int_2^3 \pi y^2\,dx$ if $xy=2$; (v) $\int_5^{10} p\,dv$ where $pv^{1\cdot 4}=2$.

5. The curve $y=4x^{\frac{1}{2}}$ is rotated about OX. Find the volume enclosed between the sections at $x=3$ and $x=8$.

R. 8

1. Fill in the following table, where A represents the area between the curve and the x-axis.

x	y	δA
3·0	5·30	
		—
3·1	4·85	
		—
3·2	4·38	
		—
3·3	3·91	

What is the approximate area between the curve and the ordinates $x=3$ and $x=3·3$ and the x-axis?

2. If G denote the gradient at a point on a curve $y=f(x)$, show that $\frac{dG}{dx}$ is negative at a point at which y is a maximum.

3. Find by Integration and also by Simpson's rule the area between $y=6x^2-x^3$ and the axis OX.

4. Find the general solution of $\frac{d^2y}{dx^2}=(2-x)^2$.

5. A cubical piece of cork of sp. gr. 0·2 and edge 10 cms. floats in water. When pushed down vertically until its base is x cms. below the surface the work (W) done is given by $\frac{dW}{dx}=$ resultant vertical force then acting on the cork. What is x when $W=0$ and what is W when $x=10$?

R. 9

1. Differentiate with respect to x:

$$\text{(i) } 2x-\frac{1}{x};\quad \text{(ii) } (3x^2+7)^3;\quad \text{(iii) } 2x^{0·6};\quad \text{(iv) } \frac{1}{4-x^2}.$$

2. The hyperbola $y^2=a+bx+cx^2$ passes through the points (0, 0), (1, 0), $(2, \sqrt{2})$. The portion between $x=1$ and $x=2$ is rotated about the x-axis; find the volume thus generated.

3. Find the area bounded by the curve $y^2=4x$, the axis Oy and the line $y=k$.

4. At what angle does the curve $y=x\left(1-\frac{x^3}{a^3}\right)$ cut the x-axis?

5. The volume V of a vessel whose height is x is given by $V=40x-2x^2$, find the area of its cross-section at the height $x=4$.

R. 10

1. In what circumstances does Simpson's rule give an exact value for an area? Find the volume of a sphere by taking only one section through the centre and by using Simpson's rule applied to volumes.

2. If V is the volume of a solid up to a section perpendicular to OX at a distance x from O, then $\frac{dV}{dx}=\pi(r^2-x^2)$. If $V=0$ when $x=-r$, find V when $x=+r$.

3. Prove that the subnormal of a curve is $y\frac{dy}{dx}$. Show that

$$\frac{d}{dx}(y^2)=2y\frac{dy}{dx},$$

and find the equation of the curve through the origin whose subnormal is constant and equal to $2a$.

4. Prove that the equation of the tangent to $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at (a, b) is $\frac{x}{a}+\frac{y}{b}=2$ for all values of n.

5. Find the mean value of the cubes of the distances from one end of the particles of a rod whose length is l.

R. 11

1. At what angle do the curves $xy=1$, $xy^2=1$ intersect at the point $(1,1)$?

2. Find the area bounded by the line $y=1$ and the part of the curve $y=x(2\frac{1}{2}-x)$ which lies above it.

3. P is a point on a semi-circle whose diameter is AB; PN is the perpendicular from P to AB, If $AB=4$, $AN=x$, $NP=y$, prove that $y^2=4x-x^2$. Hence find the position of N for which the triangle ANP is of maximum area.

4. (i) If $x=t^2$ and $y=t^2-t$, find $\frac{d^2y}{dx^2}$ in terms of t.

(ii) Evaluate $\int_{-1}^{a}(x+1)^2dx$.

5. The force P acting on a body of mass 5 lbs. is inversely proportional to the square of the distance x feet of the body from a fixed point O. If the velocity of the body v ft./sec. is given by the equation $\frac{5}{64}\frac{d}{dx}(v^2)=-P$, find the velocity of the body when it is 1 ft. from O given that the body starts from rest 2 ft. from O and that the initial value of the force is 1 lb.

CHAPTER IX

APPLICATIONS OF THE INTEGRAL CALCULUS

A. Kinematics. B. Work and Energy. C. Impulse and Momentum. D. Centre of Gravity

A. Kinematics

1. To find the distance travelled in T secs. by a body moving with variable velocity v ft./sec., v being any given function of the time t secs. measured from the starting-point.

Fig. 98 represents the velocity-time graph of the motion.

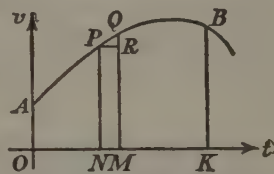

Fig. 98.

PN represents the velocity v ft./sec. of the body after time t secs. represented by ON.

Suppose the body has travelled s feet in t secs. Then with the usual notation $\delta s \simeq v \,.\, \delta t$ and so δs is represented approximately by the area of the rectangle $PNMR$.

Let OK represent T secs.

Then by dividing the area $AOKB$ into a number of rectangles such as $PNMR$ and making this number increase indefinitely we see that the area of the curve $AOKB$ represents the distance travelled in T secs.

In symbols
$$\frac{ds}{dt} = v.$$
$$\therefore\ s = \int_0^T v\,dt \qquad \text{(since } s = 0 \text{ when } t = 0).$$

2. A body starts with velocity u ft./sec. and moves with a variable acceleration a ft./sec.^{2}, where a is a given function of the time t secs.: to find its velocity after any given time T secs.

We have
$$\frac{dv}{dt}=a.$$
$$\therefore\ v=\int_0^t a\,.\,dt+c,$$
but when $t=0$, $v=u$, $\therefore\ u=0+c.$

$\therefore$ after T secs. $$v=\int_0^T a\,.\,dt+u.$$

Example 1.

The acceleration of a body after t secs. is $6t^2$ ft./sec.^{2} and its initial velocity is 3 ft./sec.; find the distance travelled in 2 secs
$$\frac{dv}{dt}=6t^2.$$
$$\therefore\ v=2t^3+c,$$
but $v=3$ when $t=0$, $\therefore\ c=3$.
$$\therefore\ \frac{ds}{dt}=v=2t^3+3,$$
$$\therefore\ s=\int_0^2(2t^3+3)\,.\,dt=\left[\tfrac{1}{2}t^4+3t\right]_0^2$$
$$=14 \text{ feet.}$$

3. *Uniformly accelerated motion.*

A body starts with velocity u ft./sec. and moving with constant acceleration a ft./sec.^{2} acquires a velocity v ft./sec. in t secs. Find v and the distance s feet it has moved.
$$\frac{dv}{dt}=a;\quad \therefore\ v=at+c,$$
since a is a constant.

But when $t=0$, $v=u$, $\therefore\ c=u.$
$$\therefore\ \underline{v=at+u.}$$

Further $$\frac{ds}{dt}=v=at+u,\quad \therefore\ s=\tfrac{1}{2}at^2+ut+b.$$

But when $t=0$, $s=0$, $\therefore\ b=0.$
$$\therefore\ \underline{s=\tfrac{1}{2}at^2+ut.}$$

4. *Uniformly accelerated rotation.*

If a body rotates about an axis and turns through θ radians in t secs., its angular velocity is measured by $\frac{d\theta}{dt}$ radians per sec.; the ordinary symbol for angular velocity is ω, so that $\omega = \frac{d\theta}{dt}$. Its angular acceleration is therefore $\frac{d\omega}{dt}$ or $\frac{d^2\theta}{dt^2}$.

If the initial angular velocity of a body is Ω and if it turns with a constant angular acceleration A, to find the angle turned through in t secs.

$$\frac{d\omega}{dt} = A, \therefore \omega = At + c.$$

But $\omega = \Omega$ when $t = 0$, $\therefore c = \Omega$.

$$\therefore \frac{d\theta}{dt} = \omega = At + \Omega.$$

$$\therefore \theta = \tfrac{1}{2}At^2 + \Omega t + b.$$

But $\theta = 0$ when $t = 0$, $\therefore b = 0$.

$$\therefore \theta = \tfrac{1}{2}At^2 + \Omega t.$$

EXAMPLES IX a

1. The velocity of a train which starts from rest is given by the following table:

t mins.	0	2	4	6	8	10	12	14	16	18	20
v mls./hr.	0	10	18	25	29	32	20	11	5	2	0

From a graph estimate the distance travelled in the 20 mins.

2. The velocity of a falling body obeys the law $v = 10 + 32t$ in ft.-sec. units. Find the distance it falls in 4 secs.

3. A body starts from rest and moves in a straight line with a speed given by $v = 16t - 4t^2$ ft.-sec. units. Find how far it goes in the third second.

4. A body starts from O with a velocity of 20 ft./sec. and moves in a straight line with an acceleration which at the end of t secs. is $(3+2t)$ ft./sec.^{2} Find its speed after 3 secs. and also the distance it travels in the 3 secs. given that t is measured from the time it leaves O.

5. A rigid body is rotating about a fixed axis with a constant angular retardation. If it turns through 100 radians in 3 secs. before coming to rest, find its initial angular velocity.

6. A body starting with a velocity of u ft./sec. moves so that its retardation is kv^2, where v is its velocity after t secs. and k is a constant. Show that $\dfrac{dt}{dv}=-\dfrac{1}{kv^2}$ and find its velocity after 1 second.

7. A body starts with a velocity of 50 ft./sec. and has a retardation of $\cdot 002v^3$ ft./sec.^{2} Find its velocity after 10 seconds.

8. A body starts from O with a velocity 6 ft./sec. and moves in a straight line so that when at a distance s feet from O its retardation is s ft./sec.^{2}; prove that $\dfrac{d}{ds}(\tfrac{1}{2}v^2)=-s$ and find its velocity when it is 3 feet from O.

9. (i) With the usual notation, prove that $\dfrac{d\omega}{dt}=\omega\,.\,\dfrac{d\omega}{d\theta}$.

(ii) A body starts to rotate about an axis with angular velocity 12 radians per sec. and its angular retardation is θ radian/sec.^{2} when the angle turned through is θ radians; find its angular velocity when it has turned through 4 radians.

10. A body is at rest at the origin O and starts to move along a curve: its acceleration parallel to the x-axis is constant and parallel to the y-axis varies as the time. What is the shape of the curve?

B. Work and Energy

If a constant force of P lbs. acts on a body and moves it through x feet in the direction of the force, then the force is said to do Px ft.-lbs. of *work*.

1. *Work done by a variable force.*

Let P be the value of the force when the distance through which it has acted upon the body is s, and let W represent the work which has then been done.

When the force acts through a further small distance δs the additional work done is given approximately by $\delta W = P\,\delta s$.

$$\therefore \frac{\delta W}{\delta s} = P \text{ approximately,}$$

and $$\frac{dW}{ds} = P \text{ exactly,}$$

$$\therefore W = \int P\,ds,$$

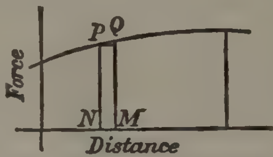

Fig. 99.

which is again represented graphically by the area of the graph in which the rectangle PM represents the element $P\,\delta s$. The important point in applying this result to problems is to remember that it is first of all necessary to find the relation between P and s, and that this relation should be found, not for their initial or final values but at some intermediate stage.

2. *Energy of a body.*

By Newton's second law of motion, if a force P lbs. acts on a body of weight w lbs., the acceleration a ft./sec.^{2} produced is given by $\frac{P}{w} = \frac{a}{g}$, where g is approximately 32.

Now with the usual notation

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$$

$$= \frac{d}{ds}\left(\tfrac{1}{2}v^2\right).$$

$$\therefore P = \frac{w}{g} \cdot \frac{d}{ds}\left(\tfrac{1}{2}v^2\right) \text{ but } \frac{dW}{ds} = P \qquad \text{(see above).}$$

$$\therefore \frac{dW}{ds} = \frac{w}{g} \cdot \frac{d}{ds}\left(\tfrac{1}{2}v^2\right).$$

$$\therefore W = \frac{w}{g}\left(\tfrac{1}{2}v^2\right) + c.$$

But if initially when $W = 0$, $v = u$ we have $0 = \frac{w}{g}\left(\tfrac{1}{2}u^2\right) + c$.

$$\therefore W = \frac{wv^2}{2g} - \frac{wu^2}{2g}.$$

The expression $\dfrac{wv^2}{2g}$ ft.-lbs. is called the *kinetic energy* of the body. The above relation shows that the work done equals the change in kinetic energy.

Extension of an elastic string or a spiral spring.

If a string or a spiral spring whose natural length is a ins. is stretched so that its length becomes $a+x$ ins., the tension T lbs. set up is given by the relation $T=\lambda . \dfrac{x}{a}$, where λ is a constant whose value depends on the nature of the spring and is called the modulus of elasticity.

If $x=a$, we see that $T=\lambda$, so that λ is the tension required to stretch the spring to twice its natural length. This relation is known as *Hooke's Law.*

3. *Work done by a couple.*

Consider a couple composed of two equal and opposite forces P acting at points A, B at a distance $2a$ apart. Then $2aP$ is called the *moment of the couple.* Let $2aP=C$. Fig. 100 represents the rotation of the couple through θ radians, the forces now acting at A', B'. Then the work done

$$= P \times \text{arc } AA' + P \times \text{arc } BB'$$

$$= Pa\theta + Pa\theta = 2Pa\theta = C\theta.$$

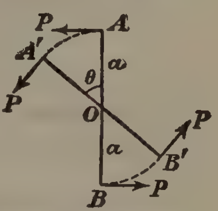

Fig. 100.

Therefore the work done by a *constant couple* (or torque) of moment C in turning through an angle of θ radians is $C\theta$.

If the couple is variable so that C is a function of θ and if W is the work done corresponding to the angle θ, then

$$\delta W \doteq C\delta\theta,$$

and

$$\frac{dW}{d\theta} = C.$$

$$\therefore \; W = \int C\, d\theta.$$

Graphically this will be represented by the corresponding area of the C, θ graph.

Example 2.

A water cart weighs $\frac{1}{2}$ ton and holds $\frac{1}{2}$ ton of water as well. It is drawn at a uniform speed of 4 mls./hr. along a level road and all the water leaks out at a uniform rate in 1 hour. Taking the coefficient of friction $\mu=0{\cdot}05$, find the work done in 1 hour in ft.-tons.

We require to find the relation between P the force required when the cart has travelled s miles. The cart travels s miles in $\frac{s}{4}$ hours and has then lost $\frac{s}{8}$ tons of water: the total weight is then $\left(1-\frac{s}{8}\right)$ tons and the friction is $\cdot 05\left(1-\frac{s}{8}\right)$ tons. Since it is travelling uniformly the necessary pull equals the friction and $P=\cdot 05\left(1-\frac{s}{8}\right)$.

$$\therefore \text{ work done} = \cdot 05\int_0^4 \left(1-\frac{s}{8}\right) ds$$

$$= \cdot 05\left[s-\frac{s^2}{16}\right] \text{ mile-tons} = \cdot 792 \text{ ft.-tons.}$$

4. *Work done in compressing a gas.*

The pressure of a gas is always stated as the force per unit area.

When the pressure is increased or diminished the volume diminishes or increases according to the law

$pv=$ const. (if the temperature remains constant),

$pv^{\gamma}=$ const. (when the expansion is adiabatic).

Suppose the gas enclosed in a cylinder and compressed by the motion of a piston whose cross-section is A. Let the volume of the gas be v when the piston is at a distance x from the end of the cylinder.

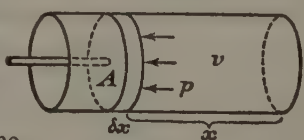

Fig. 101.

The force exerted by the gas on the piston will be Ap, and if the piston

is moved a distance δx, so small that the pressure will remain practically unaltered, then δW the work required to move the piston will be $Ap \, . \, \delta x$ (negative because δx is the *decrease* in x), i.e.

$$\delta W \triangleq -Ap \, . \, \delta x.$$

But $v = Ax$ and $\delta v = A \, . \, \delta x$.

$$\therefore \; \delta W \triangleq -p \, . \, \delta v.$$

$$\therefore \; \frac{\delta W}{\delta v} = -p \text{ and } \frac{dW}{dv} = -p.$$

$$\therefore \; W = -\int_{v_1}^{v_2} p \, . \, dv,$$

when the volume is decreased from v_1 to v_2.

Similarly the work done by the gas in expanding from v_1 to v_2, will be $\int_{v_1}^{v_2} p \, . \, dv$.

Example 3.

60 cu. ft. of air at atmospheric pressure ($14{\cdot}7 \times 144$ lbs. per sq. ft.) is compressed adiabatically until its volume is $\frac{1}{3}$ of its original volume. Find the amount of work done, given $pv^{1{\cdot}404} = k$. The work done

$$= -\int_{60}^{20} p \, dv = -\int_{60}^{20} kv^{-1{\cdot}404} \, dv$$

$$= -\left[k \, . \, \frac{v^{-0{\cdot}404}}{-0{\cdot}404}\right]_{60}^{20}.$$

But $$14{\cdot}7 \times 144 \times 60^{1{\cdot}404} = k.$$

$$\therefore \text{ work done} = +\frac{14{\cdot}7 \times 144 \times 60^{1{\cdot}404}}{{\cdot}404}\left[\frac{1}{20^{{\cdot}404}} - \frac{1}{60^{{\cdot}404}}\right]$$

$$[{\cdot}2982 - {\cdot}1913]$$

$$[{\cdot}1069]$$

$$= 176{,}000 \text{ ft.-lbs.}$$

C. Impulse and Momentum

Impulse due to a variable force.

When a constant force P acts on a moving body for time t the product Pt is known as the *Impulse*. When the force P is variable an investigation similar to B. (1), p. 116, but with t substituted for s, shows that the Impulse will be given by $\int P \, dt$.

By Newton's second law as before $\frac{P}{w}=\frac{a}{g}$, where the body is of weight w lbs.

But $a=\frac{dv}{dt}$. $\therefore \int P\,dt=\int \frac{w}{g}\times\frac{dv}{dt}\,.\,dt$

$$=\frac{w}{g}\int\frac{dv}{dt}\,.\,dt$$

$$=\frac{w}{g}\Big[v\Big]_{v_1}^{v_2},$$

where v_1, v_2 are the initial and final velocities.

$$\therefore \text{ the impulse}=\frac{w}{g}v_2-\frac{w}{g}v_1.$$

The expression $\frac{w}{g}v$ lb.-sec. units is called the *momentum* of the body. The above relation shows that the impulse equals the change in momentum.

EXAMPLES IX b

1. The pull T lbs. of a chain which is raising a weight is observed at various heights h feet and is shown in the following table:

h	0	10	15	20	25	30	35	40
T	900	850	800	720	600	500	465	450

Find the work done.

2. A body is moving horizontally under the action of a horizontal force $5x$ lbs. wt., where x is the distance in feet of the body from its starting-point O. Find the work done in moving the body 10 feet from O. Find also the velocity after moving 10 feet if the body weighs $2\frac{1}{2}$ lbs.

3. Find the work done in stretching an elastic string from 12″ to 16″ if its natural length is 6″ and if it has a length of 8″ when supporting a weight of 1 lb.

4. Find the work done in stretching a spiral spring from a length of a feet to b feet if the natural length is l feet and if a pull of p lbs. stretches it to a length of $2l$ feet.

5. A bucket of weight 3 lbs. is pulled slowly up from the bottom of a well 20 feet deep, the water in the bucket to start with weighs 20 lbs. but it leaks out at a steady rate and when the bucket arrives at the top one-fifth of its contents have dropped out. How much work has been done in drawing up the bucket?

6. A spiral spring of natural length 2 feet is such that a pull of 10 lbs. will stretch it to a length of 4 feet. How much energy is stored up in it if it is compressed to a length of 1·5 feet?

7. In winding up the spring of a gramophone, the couple required is $\frac{3}{2}+\frac{\theta}{50}$ ft.-lb. units when the handle has been turned through θ radians. How much work is needed to wind it up if it takes 20 complete turns?

8. Find the work done in compressing adiabatically 100 cu. ft. of gas at pressure 5000 lbs. per sq. ft. until its volume is 80 cu. ft., assuming that p, v obey the law $p \, . \, v^{1 \cdot 4}=\text{constant}$.

9. A cork 3″ long is drawn out slowly from a bottle; the resistance varies as the length of the cork remaining in the bottle; initially it is 50 lbs. Find the work done in extracting the cork.

10. With the data of Ex. 9, if the cork was being driven out with a steady force of 50 lbs. and if its weight was 1 oz., find the velocity with which it will leave the bottle.

11. A spiral spring of natural length 2 feet is such that a pull of 10 lbs. will stretch it to a length of 4 feet. A body of weight 8 lbs. is hooked on to the end of the spring and allowed to fall freely: (i) how much work has been done on the spring when it has stretched to a length of 3 feet; (ii) what is the velocity of the body at this moment; (iii) what is the lowest point to which the body will descend before moving upwards again?

12. The force of repulsion exerted on unit charge of electricity by a charge E is $\frac{E}{x^2}$, where x is the distance between them. Find the work done in bringing up the unit charge from an infinite distance to a point distant r from the charge E.

13. The attractive force in dynes between two magnetic poles is $\frac{m_1 m_2}{x^2}$, where m_1, m_2 represent the pole strengths and x their distance apart in cms.: find the work done in ergs (cm. dynes) in moving a pole of strength 2 from a distance of 5 cms. to 15 cms. from a pole of strength 1.

14. The force of gravitation varies inversely as the square of the distance from the centre of the earth. Find the work done in moving 1 ton from the earth's surface to a height m miles above it, taking the earth's radius as r miles.

Inside the earth the force varies directly as the distance from the centre. Find the work done in raising 1 ton to the surface from a point m miles below.

15. A variable force acts on a body initially at rest, of weight 10 lbs., such that after t secs. the force is $6-2t$ lbs. What is the impulse in the first 3 secs.? With what velocity is the body moving after 3 secs.?

16. Use Simpson's rule to find the velocity of a body of weight 4 lbs. at the end of 6 secs. if its initial velocity is 10 ft.-sec. and if it is acted on by a variable force as follows:

t (in secs.)	0	1	2	3	4	5	6
P (in lbs.)	1	1·5	2·2	3·0	3·1	2·8	2·5

17. A ship of 1000 tons goes 600 feet after steam is shut off. If the resistance of the water varies as the square of the distance the ship has to go before coming to rest, find the work done by the water to stop it given that the initial resistance is 25 lbs. wt. per ton.

18. A cube of cork, sp. gr. 0·25, whose edge is 10 cms., floats in water. Find the force required to keep its base x cms. below the surface and hence find the work done in pushing it down till just immersed.

19. An iron chain of weight 400 lbs., 20 feet long, whose sp. gr. is 8·0, hangs in water with the upper end in the surface. Find the work done in slowly lifting it vertically until its lower end is just clear of the water.

D. Centre of Gravity

The coordinates of the C. of G. of a number of particles in a plane, of weight w_1, w_2, ... at points $(x_1 y_1)$, $(x_2 y_2)$ are given by

$$\bar{x} = \frac{\Sigma wx}{\Sigma w} \text{ and } \bar{y} = \frac{\Sigma wy}{\Sigma w}.$$

The following examples will illustrate the method of applying the calculus to the determination of the C. of G. of bodies of finite size.

Example 4.

To find the C. of G. of a uniform bar AB of length $2a$. Consider a portion NM of the bar of length δx, where $AN=x$. The weight of this portion will be $w\delta x$ if w is the weight of unit length. (By taking w as the weight of unit length we can keep w constant and make the weight of NM depend on the size of δx.)

Fig. 102.

The moment of this weight about A will be between $w\delta x \cdot x$ and $w\delta x\,(x+\delta x)$.

The x-coordinate position of the C. of G. therefore lies between

$$\bar{x} \doteq \frac{\sum_0^{2a} w\delta x \cdot x}{\sum_0^{2a} w\delta x} = \frac{w\sum_0^{2a} x\,\delta x}{w\sum_0^{2a} \delta x} = \frac{\sum_0^{2a} x\,\delta x}{\sum_0^{2a} \delta x}$$

(since w is a factor of each term)

and
$$\bar{x} \doteq \frac{\sum_0^{2a} (x+\delta x)\,\delta x}{\sum_0^{2a} \delta x}.$$

At N suppose NP (y) erected so that $NP=AN$: if we do this at each point of the bar we obtain the line AC whose equation is $y=x$.

The term $x\,.\,\delta x$ is then equal to the rectangle $PN\,.\,NM$ and the term $(x+\delta x)\delta x$ is $NM\,.\,MQ$.

The expressions

$$\sum_0^{2a} x\,.\,\delta x \quad \text{and} \quad \sum_0^{2a} (x+\delta x)\,\delta x$$

are therefore represented by the sum of such rectangles as PM and QN respectively.

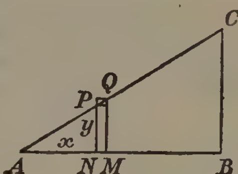

Fig. 103.

The smaller δx is taken the more nearly do our two expressions for $\bar{x}$ give the correct result, but both these series of rectangles have for their limit the area of the triangle ABC whose area is $\int_0^{2a} y\,dx$.

$$\therefore \bar{x} = \frac{\int_0^{2a} y\,dx}{\int_0^{2a} dx} = \frac{\int_0^{2a} x\,dx}{\int_0^{2a} dx} = \frac{\left[\frac{x^2}{2}\right]_0^{2a}}{\left[x\right]_0^{2a}} = \frac{2a^2}{2a} = a.$$

It will be noticed that w disappears in the working and the result is the same as if the investigation referred to a geometrical figure (here a straight line). In such cases the C. of G. is usually known as the *centroid.*

Example 5.

To find the C. *of* G. *of a right cone.*

Let the axis of the cone be OX.

Consider a section through P at right angles to the axis where $ON = x$, and let the thickness be δx.

Let w be the weight of unit volume.

Then the weight of the section is approximately $\pi y^2 \delta x . w$ and the distance of the C. of G. from O is given by

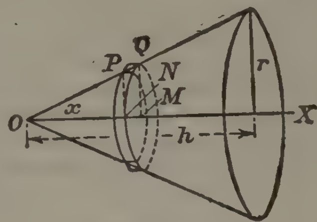

Fig. 104.

$$\bar{x} \doteq \frac{\sum_0^h w\pi y^2 \delta x . x}{\sum_0^h w\pi y^2 \delta x} \qquad (h \text{ being the height})$$

$$\doteq \frac{w\pi \sum_0^h y^2 x \,\delta x}{w\pi \sum_0^h y^2 \,\delta x} \text{ or } \frac{\sum_0^h y^2 x \,\delta x}{\sum_0^h y^2 \,\delta x}.$$

If we imagine a graph drawn such that the ordinate of a point distant x from O is Y where $Y = y^2 x$, each term of the series in the numerator for $\bar{x}$ would be represented by a rectangle $Y\,\delta x$ and the exact value of the limit of $\sum_{\delta x \to 0} y^2 x \,\delta x$ would be the area given by

$$\int Y\,dx = \int y^2 x\,dx.$$

$$\therefore \bar{x} = \frac{\int_0^h y^2 x\,dx}{\int_0^h y^2\,dx}, \text{ but } \frac{y}{x} = \frac{r}{h}.$$

$$\therefore \bar{x} = \frac{\frac{r^2}{h^2}\int_0^h x^3\,dx}{\frac{r^2}{h^2}\int_0^h x^2\,dx} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \tfrac{3}{4}h.$$

The reference to a graph to illustrate the fact that the limit of the sum of terms such as $\Sigma y^2 x \,\delta x$ is given by $\int y^2 x\,dx$ will be unnecessary in future and the fact may be assumed.

Example 6.

Find the coordinates of the centroid of a quadrant of a circle of radius r.

This example illustrates the method of finding $\bar{y}$.

The area of the strip will be $y\,\delta x$, the centroid G of this strip is approximately at a point whose ordinate is $\frac{y}{2}$.

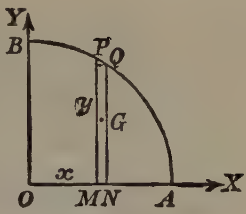

Fig. 105.

$$\therefore\ \bar{y} \doteq \frac{\sum_0^r y\,\delta x\,\frac{y}{2}}{\sum_0^r y\,\delta x} \qquad \text{but } x^2+y^2=r^2.$$

$$\therefore\ \bar{y} = \frac{\frac{1}{2}\int_0^r (r^2-x^2)\,dx}{\int_0^r \sqrt{r^2-x^2}\,dx}.$$

At present we cannot evaluate $\int_0^r \sqrt{r^2-x^2}\,dx$, but we see that here it represents the area of the quadrant of the circle and its value is therefore $\frac{1}{4}\pi r^2$.

$$\therefore\ \bar{y} = \frac{\frac{1}{2}\left[r^2x-\frac{x^3}{3}\right]_0^r}{\frac{\pi r^2}{4}} = \frac{2\left(r^3-\frac{r^3}{3}\right)}{\pi r^2} = \frac{4}{3}\,\frac{r}{\pi}.$$

By symmetry $\bar{x}=\bar{y}$, $\therefore\ \bar{x}=\frac{4r}{3\pi}$.

EXAMPLES IX c

1. A lamina is in the shape of a right-angled triangle, the sides containing the right angle being 6″, 9″. Find the distance of the centre of gravity from these two sides.

2. Find the centroid of the part of the parabola $y^2=x$ cut off by the line $x=4$.

3. Find the position of the centroid of the portion of $y=2x-x^2$ which lies above the x-axis.

4. Find the centre of gravity of a uniform hemisphere.

5. Find the centre of gravity of a pyramid whose base is a square of side 4 ins. and whose height is 6 ins., the vertex being vertically above the centre of the square.

6. The area between the curve $y^2=x$, the axis $x=0$ and the line $y=2$ is rotated about the y-axis: find the centre of gravity of the volume so obtained.

7. The portion of $y=x^2-x^3$ which lies between $x=0$ and $x=1$ is rotated about the x-axis; find the centre of gravity of the volume so obtained.

8. A body of uniform density has a rectangular base $ABCD$, $AB=3''$, $BC=4''$ and its height above any point of the base x ins. from AB is $2+x$ ins. Find (i) its volume, (ii) the height of its centre of gravity above the base.

9. A body of uniform density has a circular base of radius 3 ins. and at any point of the base r ins. from the centre of the base the height of the body is $(1+r^2)$ ins. Find (i) its volume, (ii) the height of its centre of gravity above the base.

10. Guldin's Theorem states that if any plane closed curve is revolved about an axis in its plane but not intersecting the curve, and if G is the centroid of the area of the curve, the volume of the solid so formed is measured by the product of the area of the curve and the length of the path traced out by G. Verify this theorem for the following cases:

(i) the revolution of a right-angled triangle about its shortest side;

(ii) the revolution of a quadrant of a circle about a bounding radius. Use it to find the volume of an anchor ring whose internal and external radii are a, b ins.

CHAPTER X

APPLICATIONS OF THE INTEGRAL CALCULUS

E. MOMENTS OF INERTIA. F. FLUID PRESSURE

E. Moments of Inertia

Suppose Fig. 106 represents a section of a body at right angles to an axis through O, about which the body is rotating with angular velocity ω.

Let P be the position of a particle of mass m_1 lbs. at a distance r_1 feet from O, and suppose that after time t, OP makes an angle θ with its initial position OX.

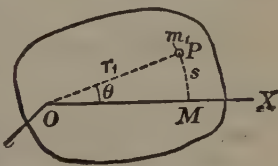

Fig. 106.

Then P is describing a circle and its speed v along the arc $= \dfrac{ds}{dt}$.

But $$s = r_1\,\theta, \quad \therefore \ \frac{ds}{dt} = r_1\frac{d\theta}{dt} = r_1\,\omega.$$

The kinetic energy of m_1 is $\dfrac{m_1 v^2}{2g} = \dfrac{m_1 r_1^{\,2}\omega^2}{2g}$ ft.-lbs.

Similarly the K.E. of a particle of mass m_2 at a distance r_2 from O is $\dfrac{m_2 r_2^{\,2}\omega^2}{2g}$ ft.-lbs.

$\therefore$ the K.E. of the whole body is $[\Sigma\,(m_1 r_1^{\,2})]\,\dfrac{\omega^2}{2g}$ ft.-lbs.

The expression Σmr^2, summed for every particle of the body, is called the *Moment of Inertia* (for short, *M.I.*) of the body about the given axis and is of great importance in the Dynamics of Rotating Bodies. When there is a continuous distribution of mass and δm is the mass of any element at a distance r from the axis, the expression becomes $\text{Lt}\,\Sigma\,(r^2\delta m)$, i.e. $\int r^2 dm$. We shall consider a few important cases.

1. *Moment of Inertia of a thin rod about an axis through its mid-point at right angles to its plane of rotation.*

Consider an element PM of length δx when $OP = x$; and $AB = 2a$. Let m be the mass of unit length of rod. Then $m\,\delta x \,.\, x^2$ is the M.I. of the element; $\therefore$ the M.I. of the whole rod

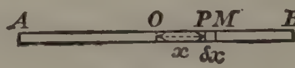

Fig. 107.

$$= \sum_{-a}^{+a} m \,.\, x^2 \,\delta x \text{ approx.}$$

$$= m \int_{-a}^{+a} x^2\,dx \quad \text{or} \quad 2m \int_0^a x^2\,dx \text{ from symmetry.}$$

$\therefore$ M.I. $= 2m\dfrac{a^3}{3} = 2am\dfrac{a^2}{3} = M\dfrac{a^2}{3}$, where $M = 2am$ is the mass of the whole rod.

N.B. Moments of Inertia are always written in the form MK^2 where M is the mass of the whole body. K is called the *radius of gyration* or *swing radius.* For the rod just considered $K = \dfrac{a}{\sqrt{3}}$, and its Moment of Inertia and K.E. would be the same as if all its mass were supposed to be concentrated at a distance of $\dfrac{a}{\sqrt{3}}$ from O.

2. *Moment of Inertia of a Rectangle about an axis in its plane bisecting a pair of opposite sides.*

Dividing the rectangle into strips parallel to the axis and taking m as the mass of unit area we obtain for the M.I.

$$2m \int_0^a 2bx^2\,dx = M\frac{a^2}{3}.$$

The expression $2\displaystyle\int_0^a 2bx^2 dx = \frac{4a^3b}{3}$ called the *second moment* of the rectangle, fig. 108, about the axis XY.

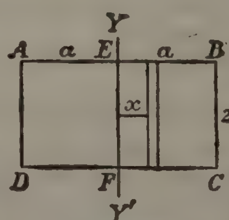

Fig. 108.

3. *Moment of Inertia of a circular disc of mass M and radius r about an axis through its centre perpendicular to its plane.*

All the material inside a ring of radius x, width δx, and height t will be the same distance from the axis.

Let m be the mass of unit volume: the area of the ring at the surface of the disc will be $2\pi x\,\delta x$ approx. and the M.I. will be

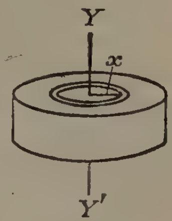

Fig. 109.

$$\sum_0^r 2\pi x\,\delta x \,.\, tm \,.\, x^2 \text{ approx.}$$

$$\therefore \text{ M.I.} = 2\pi tm \int_0^r x^3 dx = 2\pi tm \frac{r^4}{4} = M\frac{r^2}{2},$$

and $\frac{\pi r^4}{2}$ would be the *second moment* of the cross section of a disc about the same axis.

The calculation of Moments of Inertia is much simplified by the use of the two important theorems which follow.

Theorem I.

If OX, OY are two axes at right angles to one another in the plane of a *lamina* and OZ an axis at right angles to both, then

$$I_z = I_x + I_y,$$

where I_z, I_x, I_y are the M.I. about the axes OZ, OX, OY respectively.

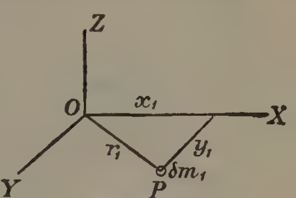

Fig. 110.

Let δm_1 be the mass of a particle at P whose coordinates are $(x_1 y_1)$; δm_2 the mass of a particle at $(x_2 y_2)$ etc.

Then
$$I_x = (\delta m_1)\, y_1^2 + (\delta m_2)\, y_2^2 + \dots,$$
$$I_y = (\delta m_1)\, x_1^2 + (\delta m_2)\, x_2^2 + \dots.$$
$$\therefore I_x + I_y = (\delta m_1)(x_1^2 + y_1^2) + (\delta m_2)(x_2^2 + y_2^2) + \dots$$
$$= (\delta m_1)\, r_1^2 + (\delta m_2)\, r_2^2 + \dots,$$

where r_1, r_2 etc. are the distances of the particles from the axis OZ.

$$\therefore I_x + I_y = I_z.$$

Theorem II.

The M.I. of a body about any axis is equal to the M.I. of the body about a *parallel axis through the* C. *of* G. together with Ma^2, where M is the mass of the body and a the distance between the two parallel axes.

Let OR be the axis and G the C. of G. of the body. Draw GO from G perpendicular to the axis OR.

Draw GZ parallel to OR, and GY perpendicular both to GO and GZ.

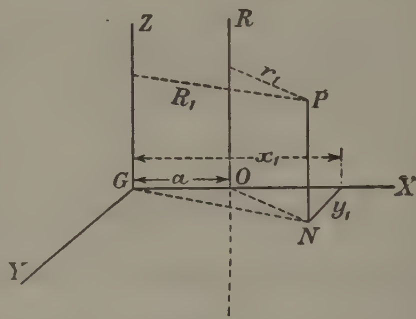

Fig. 111.

Take GO, GY, GZ as the three axes of coordinates and let the coordinates of a particle δm_1 at P be x_1, y_1, z_1: r_1 its distance from the axis OR and R_1 its distance from GZ.

Then M.I. of the body about OR

$$= (\delta m_1)\, r_1^2 + (\delta m_2)\, r_2^2 + \dots$$

$$= \delta m_1 \left[(x_1 - a)^2 + y_1^2\right] + \delta m_2 \left[(x_2 - a)^2 + y_2^2\right] + \dots$$

$$= \delta m_1 (x_1^2 + y_1^2) + \delta m_2 (x_2^2 + y_2^2) + \dots$$

$$+ a^2 \left[\delta m_1 + \delta m_2 + \dots\right] - 2a \left[(\delta m_1)\, x_1 + (\delta m_2)\, x_2 + \dots\right].$$

Now the X coordinate of the C. of G. of the body is given by $\bar{x} = \dfrac{\Sigma\left[(\delta m)\, x\right]}{M}$, but here $\bar{x} = 0$.

$$\therefore\ (\delta m_1)\, x_1 + \delta m_2 (x_2) + \dots = 0,$$

$$\therefore\ \text{M.I.} = (\delta m_1)\, R_1^2 + (\delta m_2)\, R_2^2 + \dots$$

$$+ a^2 . M$$

$$= \underline{I_G + Ma^2},$$

where I_G is the M.I. of the axis through G parallel to OR. We shall use these theorems to obtain some further results.

4. *The* M.I. *of a rectangle 2a by 2b about an axis through its mid-point perpendicular to its plane is*

$$M\frac{a^2+b^2}{3}.$$

For M.I. about $EF = M\dfrac{a^2}{3}$, and about GH it equals $M\dfrac{b^2}{3}$.

$\therefore$ since $I_z = I_x + I_y$ we have

$$I_z = M\frac{a^2+b^2}{3}.$$

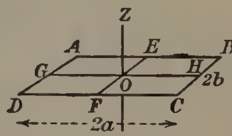

Fig. 112.

5. *The* M.I. *of a circular disc about a diameter is* $M\dfrac{r^2}{4}$.

$I_y = I_x$ by symmetry. But $I_z = M\dfrac{r^2}{2}$. (See p. 129.)

$$\therefore M\frac{r^2}{2} = 2I_x.$$

$$\therefore I_x = I_y = M\frac{r^2}{4}.$$

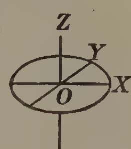

Fig. 113.

6. *The* M.I. *of a sphere of mass M about a diameter is* $M\dfrac{2r^2}{5}$.

Take OX as the diameter and divide the sphere into elements by planes perpendicular to OX. Consider the section at P (x, y) whose thickness is δx; this is approximately a circular disc of volume $\pi y^2\,\delta x$ and mass $m\pi y^2\,\delta x$ if m is the mass of unit volume. The M.I. of this disc about OX $\left(M\dfrac{r^2}{2}\right)$ will be

$$\doteq m\pi y^2 \delta x \frac{y^2}{2}.$$

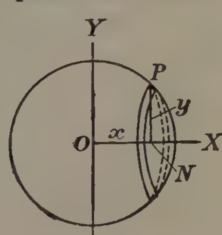

Fig. 114.

$\therefore$ the M.I. of the sphere $= \frac{m\pi}{2}\int_{-r}^{+r} y^4\,dx$ or (since it is symmetrical about the plane through O perpendicular to OX)

$$= 2\,\frac{m\pi}{2}\int_0^r y^4\,dx$$

$$= m\pi\int_0^r (r^2-x^2)^2\,dx$$

$$= m\pi\int_0^r (r^4-2r^2x^2+x^4)\,dx$$

$$= m\pi\,\frac{8r^5}{15}$$

$$= M\,\frac{2r^2}{5} \text{ since } M = m\,.\,\tfrac{4}{3}\pi r^3.$$

7. *The* M.I. *of a circular disc of radius r about a tangent.*

The M.I. about a diameter through $G = M\,\frac{r^2}{4}$.

$\therefore$ by parallel axis theorem, the M.I. about a parallel axis through A is

$$M\,\frac{r^2}{4} + Mr^2 = M\,\frac{5r^2}{4}.$$

Fig. 115.

8. *Find the* M.I. *of a solid cylinder of mass M, radius r, and length l about a diameter of one end.*

Divide the cylinder into circular discs parallel to the top: and consider a disc at distance x from AB.

The mass of the disc will be $\pi r^2\,\delta x\,.\,m$, and its M.I. about a diameter $A'B'$ will be

$$\left(M\,\frac{r^2}{4}\right) = \pi r^2\,\delta x\,.\,m\left(\frac{r^2}{4}\right).$$

$\therefore$ by Theorem II its M.I. about AB will be

$$\pi r^2\,\delta x\,.\,m\,\frac{r^2}{4} + \pi r^2\,\delta x\,.\,mx^2.$$

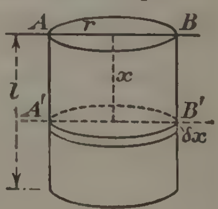

Fig. 116.

∴ M.I. of cylinder about AB

$$= \pi r^2 m \int_0^l \left(\frac{r^2}{4} + x^2\right) dx$$

$$= \pi r^2 m \left[\frac{r^2 l}{4} + \frac{l^3}{3}\right]$$

$$= \frac{\pi r^2 ml}{12}(3r^2 + 4l^2)$$

$$= \frac{M}{12}(3r^2 + 4l^2).$$

EXAMPLES X a

1. Find the radius of gyration of a thin rod 4 feet long about an axis at one end perpendicular to the rod.

2. Find the M. I. of a uniform flywheel weighing 120 lbs. whose radius is $1\frac{1}{2}$ feet about an axis through its centre perpendicular to its plane.

3. Find the M. I. about its axis of a hollow right cylinder, 2 feet high, whose diameter inside is 8 inches and outside 10 inches, given that its mass is 100 lbs.

4. Find the radius of gyration of a rectangular plate 4 feet by 2 feet and thickness 3 inches about an axis perpendicular to its largest face at its centre.

5. A circular door of a furnace is hinged so that it turns about a tangent. If its diameter is 18 inches and mass 60 lbs., find its M. I. and its radius of gyration.

6. Find the K. E. of a uniform flywheel of mass 100 lbs., whose radius is 8 inches, when making 6 revolutions per sec. about its axis.

7. Find the M. I. of a uniform solid sphere, radius a feet, mass m lbs., about any tangent line.

8. A uniform rod 12 inches long of weight 4 lbs. is pivoted about a point 2 inches from its centre. Find its kinetic energy when it is making 5 revolutions per sec. about this point.

9. Find the M. I. of a uniform solid right circular cone, base-radius r inches, height h inches, mass m lbs., about its axis.

10. Find the radius of gyration of a square of side a inches about one diagonal.

11. A lamina is in the shape of the segment cut off from $y^2 = 4ax$ by the double ordinate $x = b$; its mass is m lbs.; find its M.I. about the tangent at the vertex, i.e. the y-axis.

12. A solid cylinder of radius 6 inches and length 8 inches is rolling along a plane at the rate of 2 revolutions a second. Its mass is 100 lbs. Find (i) its M.I. about the line in contact with the plane at any moment, (ii) its kinetic energy.

Routh's Rule

For practical purposes Moments of Inertia are generally remembered by a mnemonic known as Routh's Rule which states that for any one of three mutually perpendicular axes of symmetry through the C. of G., the M.I. is equal to $M \times$ sum of squares of the other two semi-axes divided by 3 for a rectangle, cube or cuboid, 4 for a circle or ellipse, 5 for a sphere or spheroid.

For a thin rod AB of length $2a$ about an axis perpendicular to AB at its mid-point the rule gives

$$M \times \frac{a^2 + 0^2}{3} = M \frac{a^2}{3}.$$

For a cuboid with edges $2a$, $2b$ about an axis perpendicular to both at the mid-point of the face we have

$$M \times \frac{a^2 + b^2}{3}.$$

For a circular plate of radius r, about an axis perpendicular to its plane at the centre, we have

$$M \times \frac{r^2 + r^2}{4} = M \frac{r^2}{2}.$$

About a diameter we have

$$M \times \frac{r^2 + 0^2}{4} = M \frac{r^2}{4}.$$

For a sphere about a diameter

$$\text{M.I.} = M \times \frac{r^2 + r^2}{5} = M \frac{2r^2}{5}.$$

F. Fluid Pressure

The following laws of Hydrostatics will be assumed:

(i) The pressure of a fluid against a plane surface in contact with it is at right angles to that surface.

(ii) The pressure at a point in a fluid is the same in all directions.

(iii) The pressure is proportional to the depth below the free surface.

I. *Total pressure on a plane surface.*

To find the pressure on a vertical triangular lamina immersed in a fluid with its base in the surface.

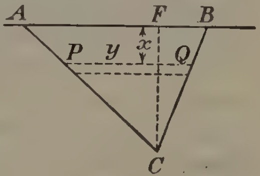

Fig. 117.

Let the base $AB = b$ inches, the altitude $CF = h$ inches and let 1 cu. in. of the fluid weigh w lbs.

Consider a strip PQ of length y inches parallel to AB and at vertical depth x inches below AB; the width of the strip is δx inches.

The pressure at right angles to the strip is the same as though the strip were horizontal at the same depth, in which case it would be supporting approximately a column of fluid of volume $y\,\delta x x$ cu. ins. and weight $wxy\,\delta x$ lbs.

$\therefore$ the pressure on the strip $PQ \doteq wxy\,\delta x$ lbs.

$\therefore$ the total pressure on the lamina $\doteq \sum\limits_{0}^{h} wxy\,\delta x$ lbs.

$\therefore$ the total pressure $= w\int_0^h xy\,.\,dx$ since w is a constant. From similar triangles $\dfrac{y}{b} = \dfrac{h-x}{h}$,

$$\therefore\ y = \frac{b}{h}(h-x).$$

$$\therefore\ \text{pressure} = \frac{wb}{h}\int_0^h x(h-x)\,dx = \frac{wb}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_0^h$$

$$= \frac{wb}{h} \times \frac{h^3}{6} = \frac{wbh^2}{6}\ \text{lbs.}$$

Note. (i) If the liquid is water, the value of w is approx. 0·0361.

(ii) If the dimensions are in cms. and if the pressure is measured in grs., $w=1$.

(iii) If the liquid is of specific gravity s, the value of w is s times the amount it would be for water.

(iv) Since the area of the lamina $=\frac{1}{2}bh$ and since the depth of the centre of gravity $=\frac{h}{3}$, we see that the pressure on the lamina equals the weight of the column of fluid supported by the lamina when at a depth equal to that of its centre of gravity. We shall now prove that this is true for any lamina.

To find the pressure on any plane lamina immersed in a fluid with its plane inclined at an angle θ to the vertical.

Consider a strip parallel to the line of intersection BC of the plane of the lamina with the surface and let its distance measured from BC in the plane of the lamina be x.

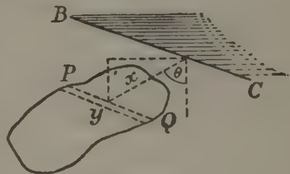

Fig. 118.

Its area is $y\,\delta x$ and its depth below the surface is $x\cos\theta$.

$\therefore$ pressure upon it $\doteq y\,\delta x\,(x\cos\theta)\,w$.

The total pressure on the lamina

$$= w\cos\theta\,\Sigma xy\,\delta x \text{ approx.}$$

$$= w\cos\theta \int xy\,dx \text{ exactly.}$$

Now if $\bar{x}$ is the distance of the C. of G. of the lamina from BC we have

$$\bar{x} = \frac{\Sigma(y\,\delta x)\,x}{\Sigma y\,\delta x} \text{ approx.}$$

$$= \frac{\int xy\,dx}{\int y\,dx} \text{ exactly.}$$

$$\therefore\ \bar{x}\int y\,dx = \int xy\,dx.$$

$\therefore$ total pressure $= w\cos\theta\,\bar{x}\int y\,dx = w\cos\theta\,\bar{x}\,A$

where A is the area of the lamina

$$= wA\,(\bar{x}\cos\theta)$$

$$= wA\,(\text{depth of C. of G.}).$$

The pressure is therefore the same as if the whole lamina were immersed at the depth of its C. of G.

II. *Centre of Pressure.*

If a plane surface is subject to fluid pressure, the point of intersection of the resultant pressure with the plane is called the *centre of pressure.*

To find the C. of P. of a triangular lamina whose base is in the surface and whose plane is inclined at an angle θ to the vertical.

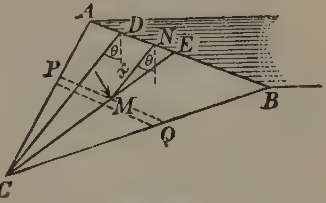

Fig. 119.

Draw CD perpendicular to AB and draw the median CE.

Divide the lamina into strips (PQ), parallel to the surface, at a distance x from AB, then the width of each will be δx.

The resultant pressure on this strip will act at its mid-point, i.e. on the median CE.

The pressure on the strip will be $(y\,\delta x)(x\cos\theta)\,w$ and since this acts at right angles to the lamina its moment about AB will be

$$(y\,\delta x)(x\cos\theta)\,w\,.\,MN = w\cos\theta\,x^2y\,\delta x.$$

If $\bar{x}$ is the distance from AB (along the lamina) of the point of action of the resultant we have, by taking moments about AB,

$$\bar{x} \times \text{total pressure} = \text{sum of the moments about } AB,$$

$$\text{i.e. } \bar{x}\sum_0^h y\,\delta x\,.\,x\cos\theta w \doteq \sum_0^h w\cos\theta\,x^2y\,\delta x \text{ when } DC = h.$$

$$\therefore\ \bar{x} = \frac{w\cos\theta\int_0^h x^2y\,dx}{w\cos\theta\int_0^h xy\,dx}$$

$$= \frac{\int_0^h x^2y\,dx}{\int_0^h xy\,dx}.$$

Here $\frac{y}{b} = \frac{h-x}{h}$, and on simplification we get $\bar{x} = \frac{h}{2}$.

Since then $RS = \frac{CD}{2}$,

$$\therefore RE = \frac{CE}{2}.$$

$\therefore$ the C. of P. is at the mid-point of the median.

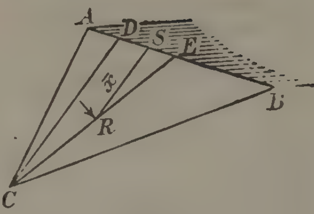

Fig. 120.

Note. (i) It is important to notice that this result is independent of θ so that as the angle of inclination to the vertical alters the position of the C. of P. in the lamina remains unchanged, but it must be remembered that the total pressure on the lamina obviously alters.

(ii) The condition that the lamina was triangular was not introduced until we had proved that $\bar{x} = \frac{\int x^2 y\, dx}{\int xy\, dx}$, so that this formula is of general application.

Example.

To find the C. of P. of a rectangle $ABCD$ immersed with one edge AB parallel to the surface at a depth h.

Let $\bar{x}$ be the depth of the C. of P. below AB. Take moments about AB, then as before we have

$$\bar{x}\, \Sigma a\, \delta x\, (x+h)\, w \doteq \Sigma a\, \delta x\, (x+h)\, w \, . \, x,$$

from which we obtain $\bar{x} = \frac{b\,(3h+2b)}{3\,(2h+b)}$.

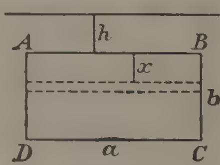

Fig. 121.

Alternative method. The C. of P. of the rectangle when the upper edge lies in the surface will be found to be at a distance $\frac{2}{3}b$ below it, and the pressure is $ab\left(\frac{b}{2}\right)$ units.

The effect of immersing it a further depth h is to add the same increase of pressure to each unit of area, i.e. to add a total force of $(ab)\,h$ units acting at the C. of G. of the lamina. The resultant R of these two forces will be $(ab)\,h + ab\left(\frac{b}{2}\right)$ acting at a distance $\bar{x}$ from the upper edge and by taking moments about

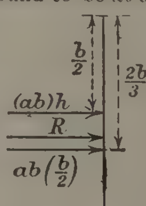

Fig. 122.

this edge we have $ab\left[h+\frac{b}{2}\right]\bar{x}=(ab)\,h\,.\left(\frac{b}{2}\right)+ab\left(\frac{b}{2}\right).\frac{2b}{3}.$

$$\therefore\ \bar{x}=\frac{(3h+2b)\,b}{3\,(2h+b)}.$$

We have hitherto neglected the effect of the atmospheric pressure at the surface of the liquid. If this is taken into account we can either increase the pressure on each element by the weight of a column of liquid equivalent to the atmospheric pressure or find the effect of this additional force by the alternative method of the previous example.

EXAMPLES X b

Work out from first principles, taking the weight of 1 cu. ft. of water as 62·3 lbs.:

1. Find the pressure in lbs. weight on a triangular lamina immersed vertically in water with its apex in the surface and base horizontal if its height is h feet and base b feet.

2. Find the C. of P. of the lamina in Ex. 1.

3. A triangle having a base of 24 inches and each of the other sides 36 inches is immersed vertically in water with its base in the surface; calculate the depth of the centre of pressure.

4. A rectangular vessel containing gas has a rectangular lid 6 inches by 4 inches hinged at the long edge. If the gas pressure is 20 lbs./sq. in., find the moment of this pressure about the hinge. If the air pressure is 15 lbs./sq. in., find the force at the opposite edge required to keep the lid closed, if the weight of the lid is neglected.

5. A sluice gate is 6 feet wide and 8 feet deep; find the total pressure on one side when the water is up to the top. Find also the position of the C. of P.

6. The end of a trough is an isosceles triangle 2 feet base and altitude 1 foot 6 inches, inclined at an angle of 60° to the horizontal. Find the pressure on it and the centre of pressure when the trough is full of water.

7. A lock gate is 35 feet wide and the depth of the water on the two sides is 26 feet and 13 feet respectively. Find the resultant water pressure and the distance from the bottom of the gate at which it acts. Does the atmospheric pressure affect the answer?

8. If a triangle whose height is 10 cms. and base 20 cms. be immersed with vertex downwards in a liquid of density 1·5 gms./c.c. with its base horizontal and 20 cms. below the surface, find the pressure on one side and the C. of P. including a pressure at the free surface of 1 kg./sq. cm.

9. A drain is closed by a square door of side 3 feet hinged at its upper edge and inclined at 45° to the horizontal. Find its weight if it is just to open when the drain is half full of water.

10. Show that the depth of the C. of P. of a surface measured along its plane is given by $y=\dfrac{\text{Second Moment of the Area}}{\text{Moment of Area}}$ both taken about the axis of floatation (the line in which the plane of the lamina cuts the surface).

Hence find the C. of P. of a circular plate placed vertically and just touching the surface, if its diameter is 3 feet.

11. A plane area S is immersed vertically in a liquid so that the depth of the centroid G of the area is h; show that the depth of the C. of P. is given by $y=h+\dfrac{\text{2nd Moment}}{Sh}$, where the 2nd Moment is about the horizontal axis through G. What is the limiting position of the C. of P. when the depth is indefinitely increased?

12. A square of side 1 foot is placed in water with a diagonal vertical and one end of the diagonal in the surface.

Find the depth of the centre of pressure.

MISCELLANEOUS EXAMPLES 7—12

M. 7

1. Sketch the curve $y^2=x^2-x^4$ and find the volume of the solid obtained by rotating it about the x-axis.

2. In $\Delta\, ABC$, $AB=AC=6$ ins., $\angle BAC=90°$; PQR is an isosceles right-angled triangle of variable size with its hypotenuse PQ parallel to BC. What is the greatest area common to the two triangles?

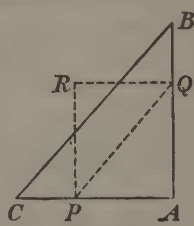

Fig. 123.

3. Sketch the curve $x=at^2$, $y=bt^3$, where a, b are constants and t is a variable parameter. If the tangent at any point P on the curve meets the curve again at Q, prove that PQ is divided by the x-axis in the ratio 8 : 1.

4. From a square of side 6 inches a square of side 2 inches is cut out at one of the corners; find the moment of inertia of the lamina about an axis perpendicular to its plane through the opposite corner, given that the mass of the lamina is 8 lbs.

5. A body starts to rotate about an axis with angular velocity 10 radians per sec. and its angular retardation is 4θ radians/sec.^{2} when the angle turned through is θ radians; find its angular velocity when it has turned through 4 radians and the total angle through which it turns before coming to rest.

M. 8

1. AB is a line of length 20 ins.; a circle of radius 8 ins. touches AB at B; it starts to roll along AB towards A and at the same moment a particle P starts from A and moves with equal speed along a line perpendicular to AB. Find the length of AP when the circle appears to look largest from P, i.e. when the tangent from P to the circle is least.

2. The area bounded by the curve $y=x^2$ between the axis $x=0$ and the line $y=1$ is rotated about the y-axis to form a solid. Find the position of its centre of gravity.

3. A uniform circular disc of mass 10 lbs., radius 8 ins., is rotating about a point 2 ins. from the centre at the rate of 20 revolutions per minute. Find its kinetic energy.

4. A beam AB of uniform section is such that the weight of any portion $AP=\frac{1}{2}x+\frac{1}{5}x^2$ lbs., where $AP=x$ feet. Compare the densities at the mid-point of AB and at B, if $AB=5$ feet.

5. A body is at rest at the origin O and starts to move along a curve so that its acceleration parallel to the x-axis varies as the time and parallel to the y-axis varies as the square of the time. Find the shape of the curve.

M. 9

1. Prove that $\int_0^2 x(x-1)(x-2)\,dx=0$ and explain this result. Find the values of x, giving the maximum and minimum values of

$$x(x-1)(x-2).$$

2. A line is drawn through the vertex of the parabola $y^2=4x$ inclined at 45° to OX. Find the area between the curve and the line.

3. A portion of a sphere of radius a is cut off by a plane at a distance $\frac{a}{2}$ from the centre. Show that the volume of this part is $\frac{5}{32}$ of the whole sphere.

4. Water is admitted to a dock whose cross-section is a parabola of depth 30 ft. and width at the top 40 ft. Find the water pressure on the vertical end when the depth is 20 ft. and the rate at which it is increasing if the water is then rising at 4 ins. per min.

5. Find the ratio of the M. I. of two areas, one a square of side a and the other an I-shape of equal area, about the dotted axis. (Army.)

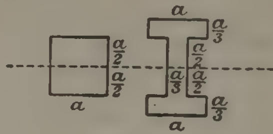

Fig. 124.

M. 10

1. The equation of a curve is $y=(a+bx)^{-n}$. From $A(x_1, y_1)$ and $B(x_2, y_2)$ two points on it, AD and BC are drawn perpendicular to OX. Find the area $ABCD$. Taking $a=0$, show that the area reduces to $\dfrac{x_1y_1-x_2y_2}{n-1}$ if $n>1$. Evaluate when $y_1=7{\cdot}5$; $y_2=1{\cdot}0$; $n=3$; $b=1$.

2. Find the volume of a gasholder which consists of a cylinder 30 ft. in diameter and 20 ft. high, surmounted by a spherical segment 10 ft. high.

3. When air expands adiabatically the relation between P the pressure (lbs./sq. in.) and V the volume in cu. in. is $PV^{1{\cdot}408}=$ const. Find the work done if the volume and pressure change from P_1 and V_1 to P_2 and V_2.

If the relation between the pressure, volume and absolute temperature T is $\dfrac{PV}{T}=1170$, find the volume in cu. ft. occupied by 1 lb. of air when $P=90$ lbs./sq. in. and $T=521$. Find the work done when this volume expands adiabatically until its pressure is 15 lbs./sq. in.

4. The small tank in Fig. 125 is full of water. $ABEF$ is vertical and $ABCD$ is horizontal. Find the resultant thrust on the base $DCEF$ and the point where it acts.

$$AD=BC=96 \text{ cms.};$$
$$DC=AB=FE=20 \text{ cms.};$$
$$AF=BE=28 \text{ cms.}$$

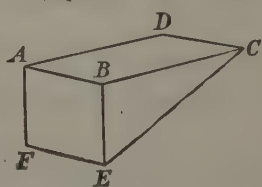

Fig. 125.

5. Find the maximum and minimum values and the least value of the function of x which is equal to $2x(x-1)$ when $x<1$, and is equal to $(x-1)(x-2)(x-3)$ when $x>1$. Is the slope of the graph continuous? (Cambridge University.)

M. 11

1. Verify that the curve

$$2a^2y = h_1(x-a)(x-2a) - 2h_2x(x-2a) + h_3x(x-a)$$

goes through $A(0, h_1)$, $B(a, h_2)$, $C(2a, h_3)$, and prove that the area between ABC, OX, and the ordinates at A and C is $\frac{a}{3}(h_1+4h_2+h_3)$ provided the curve does not cross OX when x is between 0 and $2a$. Prove that the tangent at B is parallel to the chord AC. (Army.)

2. Find the radius of gyration of a uniform solid hemisphere about a diameter of the base.

3. A cylindrical hole of diameter $2c$ is drilled through a solid sphere of diameter $2a$, the axis of the cylinder passing through the centre of the sphere. Show that the volume of the remaining portion of the sphere is $\frac{4}{3}\pi(a^2-c^2)^{\frac{3}{2}}$. Show that this is equal to $\frac{\pi}{6}h^3$, where h is the length of the axis of the cylinder. (Cambridge University.)

4. A rectangular trough 5 feet deep and 3 feet broad is closed by a heavy sluice gate hinged at its upper edge and resting on the bottom of the trough at an angle of 60° to the horizontal. Find the weight of the gate if it is just to open when the trough is full of water.

5. When water escapes through a sharp-edged orifice in the bottom of a tank the number of cubic feet which passes per sec. is given by $Q = Ca\sqrt{2gh}$, where a is the cross-section of the orifice in square feet, h the head of water above the hole in feet and C is a constant. Obtain an expression for the time required to lower the level of the water in a cylindrical tank of A sq. ft. cross-section from a depth H_1 ft. to H_2 ft.

Evaluate when the diameter of the tank is 10 ft.; diameter of orifice, 3 ins. $H_1=25$, $H_2=5$, $C=0{\cdot}60$, $g=32$. (Army.)

M. 12

1. The explosive head of a torpedo is formed by the revolution about OX of that half of the ellipse $\frac{x^2}{4}+y^2=81$ for which x is positive, the unit for each axis being 1 in. Find the volume of the head.

2. The section of a river, maximum depth 15 feet, is such that its breadth x feet below the surface is $40-\frac{x^2}{15}$ feet. Assuming that the velocity decreases uniformly from v ins. per sec. at the surface to

$v-2\sqrt{v}+1$ ins./sec. at the bottom, find the number of cubic feet of water that pass a given point on the bank per sec. when the surface velocity is 16 ins. per sec.

3. Draw a rough graph of $y=\dfrac{x^2-4x+9}{x^2+4x+9}$.

4. A wall-sided barge is 30 ins. deeper in the water when loaded with 50 tons of coal than when empty. Calculate the area enclosed by the water line.

The interior of the barge is rectangular in shape: the 50 tons of coal fill this space to a depth of 4 ft. and when the barge is thus loaded the horizontal floor of this space is 3 ft. below the water level. The coal is raised from the barge to a height of 5 ft. above the water level. Find expressions for the lowering of the coal surface relative to the barge when x tons have been unloaded and the height through which the coal is then being raised and deduce the work done in raising the whole load of coal in ft.-tons. (Army.)

5. A spiral spring is stretched slowly: if the total extension is 5 ins. and the final value of the tension 40 lbs., find the work done in stretching it from its natural position.

CALCULUS FOR SCHOOLS

PART II

CHAPTER XI

GENERAL METHODS OF DIFFERENTIATION

I. If u is any function of x and if C is a constant,

$$\frac{d}{dx}(Cu) = C\frac{du}{dx}.$$

When any increment δx is made in x, the corresponding increment in u is denoted by δu.

$\therefore$ if $y = Cu$, $\quad y + \delta y = C(u + \delta u)$,

$$\therefore \delta y = C(u + \delta u) - Cu = C \,.\, \delta u,$$

$$\therefore \frac{\delta y}{\delta x} = C\frac{\delta u}{\delta x}.$$

$\therefore$ in the limit when $\delta x \to 0$

$$\frac{d}{dx}(Cu) = \frac{dy}{dx} = C\frac{du}{dx}.$$

Example 1.

$$\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

II. If u, v, w are any functions of x,

$$\frac{d}{dx}(u + v + w + \ldots) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots.$$

Let $y = u + v + w + \ldots$ and when any increment δx is made in x, let the corresponding increments in $u, v, w \ldots y$ be denoted by $\delta u, \delta v, \delta w \ldots \delta y$.

Then
$$y + \delta y = (u + \delta u) + (v + \delta v) + \ldots,$$

$$\therefore \delta y = \delta u + \delta v + \delta w + \ldots,$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x} + \ldots. \qquad 6$$

$\therefore$ in the limit when $\delta x \to 0$

$$\frac{d}{dx}(u+v+w+\ldots)=\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots.$$

Note that in the same way we may show that

$$\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}.$$

Example 2.

Find $\frac{dy}{dx}$ when $y=3x^2-5x+4$.

$$\frac{dy}{dx}=\frac{d(3x^2)}{dx}-\frac{d(5x)}{dx}+\frac{d(4)}{dx}=6x-5.$$

Note, if u, x, y, z are all functions of t and $u=x+y+z$,

$$\frac{du}{dt}=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}.$$

III. *Differentiation of a Product.*

If u, v are any functions of x,

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}.$$

Let $y=uv$, then with the same notation as in II. we have

$$y+\delta y=(u+\delta u)(v+\delta v),$$

$$\therefore \delta y=(u+\delta u)(v+\delta v)-uv$$

$$=u\,.\,\delta v+v\,.\,\delta u+\delta u\,.\,\delta v.$$

$$\therefore \frac{\delta y}{\delta x}=u\,.\,\frac{\delta v}{\delta x}+v\,.\,\frac{\delta u}{\delta x}+\frac{\delta u}{\delta x}\,.\,\frac{\delta v}{\delta x}\,.\,\delta x.$$

When $\delta x \to 0$ we have $\frac{dy}{dx}=u\,.\,\frac{dv}{dx}+v\,.\,\frac{du}{dx}.$

Example 3.

Find $\frac{d}{dx}\{(x^2+1)(2x+3)\}$.

$$\frac{d}{dx}\{(x^2+1)(2x+3)\}=(x^2+1)\,.\,\frac{d}{dx}(2x+3)+(2x+3)\,.\,\frac{d}{dx}(x^2+1)$$

$$=(x^2+1)\,.\,2+(2x+3)\,.\,2x$$

$$=2x^2+2+4x^2+6x$$

$$=6x^2+6x+2.$$

As a check, $(x^2+1)(2x+3)=2x^3+3x^2+2x+3,$

$$\therefore \frac{d}{dx}\{(x^2+1)(2x+3)\}=6x^2+6x+2.$$

This result may be extended to a product consisting of more than two factors.

Thus
$$\frac{d}{dx}(uvw) = \frac{d}{dx}[(uv)(w)]$$
$$= uv \cdot \frac{dw}{dx} + w\frac{d}{dx}(uv)$$
$$= uv \cdot \frac{dw}{dx} + w\left[u\frac{dv}{dx} + v\frac{du}{dx}\right]$$
$$= uv \cdot \frac{dw}{dx} + wu\frac{dv}{dx} + vw\frac{du}{dx}.$$

A useful form of this result which shows its extension to any number of factors is obtained by writing it in the form:

$$\frac{\frac{d}{dx}(uvw)}{uvw} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx}.$$

IV. *Differentiation of a Quotient.*

If u, v are any functions of x,

$$\mathbf{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}.$$

With the same notation:

If $y = \frac{u}{v}$, $\qquad y + \delta y = \frac{u + \delta u}{v + \delta v}$;

$$\therefore \ \delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$
$$= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$$
$$= \frac{vu + v \cdot \delta u - uv - u \cdot \delta v}{v(v + \delta v)};$$
$$\therefore \ \frac{\delta y}{\delta x} = \frac{v \cdot \frac{\delta u}{\delta x} - u \cdot \frac{\delta v}{\delta x}}{v^2 + v \cdot \frac{\delta v}{\delta x} \cdot \delta x}.$$

$\therefore$ when $\delta x \to 0$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 4.

Find $\dfrac{d}{dx}\left(\dfrac{x^2+1}{3x+2}\right)$.

$$\frac{d}{dx}\left(\frac{x^2+1}{3x+2}\right) = \frac{(3x+2)\dfrac{d}{dx}(x^2+1) - (x^2+1)\dfrac{d}{dx}(3x+2)}{(3x+2)^2}$$

$$= \frac{(3x+2)(2x) - (x^2+1)(3)}{(3x+2)^2} = \frac{6x^2+4x-3x^2-3}{(3x+2)^2}$$

$$= \frac{3x^2+4x-3}{(3x+2)^2}.$$

V. $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}}$.

Now $$\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1.$$

But when δx and $\delta y \to 0$ the limit of $\dfrac{\delta y}{\delta x}$ is $\dfrac{dy}{dx}$ and the limit of $\dfrac{\delta x}{\delta y}$ is $\dfrac{dx}{dy}$.

$$\therefore \frac{dx}{dy} \times \frac{dy}{dx} = 1,$$

i.e. $$\frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}}.$$

This result may be deduced as a particular case of VI. Since

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},$$

we have if y is put equal to x

$$\frac{dx}{dx} = \frac{dx}{du} \times \frac{du}{dx},$$

i.e. $$1 = \frac{dx}{du} \times \frac{du}{dx} \text{ or } \frac{dx}{du} = \frac{1}{\dfrac{du}{dx}}.$$

This result is also evident from geometrical considerations.

In each of the figures $\frac{dy}{dx} = \tan\psi$, $\frac{dx}{dy} = \tan\psi'$.

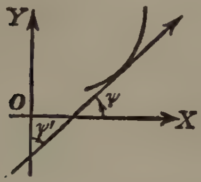

Fig. 126.

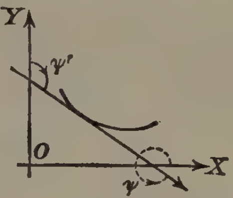

Fig. 127.

In Fig. 126, $\psi + \psi' = \frac{\pi}{2}$ and in Fig. 127, $\psi + \psi' = \frac{5\pi}{2}$,

$$\therefore \text{ in each case } \tan\psi' = \cot\psi \text{ or } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

It should be noted that the sense of the tangent as indicated by the arrow is that in which x is increasing: further, the angle which a line of definite sense makes with $\overrightarrow{OX}$ is measured by the anti-clockwise rotation from OX *towards* OY and the angle it makes with $\overrightarrow{OY}$ by the clockwise rotation from OY *towards* OX

Example 5.

If $y = x^{\frac{1}{3}}$, find $\frac{dy}{dx}$.

Now $x = y^3$, $\quad \therefore \frac{dx}{dy} = 3y^2 = 3x^{\frac{2}{3}}$

$$\therefore \frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

VI. *Function of a function.*

We shall illustrate the method by an example. It consists in the application of the formula $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$.

Example 6.

To find $\dfrac{dy}{dx}$ when $y=(ax+b)^{10}$.

Let $u=(ax+b)$, then $y=u^{10}$.

y is here a function of u and u is a function of x; $\therefore$ y is a function of a function of x.

Now $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$, but $\dfrac{dy}{du}=10u^9$ and $\dfrac{du}{dx}=a$;

$$\therefore\ \frac{dy}{dx}=10u^9(a)=10a(ax+b)^9.$$

Note. $\dfrac{d}{dx}(ax+b)^{10}$ is *not* $10(ax+b)^9$ but $10a(ax+b)^9$.

This substitution method is of general application to functions of a function.

EXAMPLES XI a

Differentiate with respect to x:

1. $5x^3+2x-4$. **2.** $4x^5-5x^4$. **3.** $2\frac{1}{2}x^6-1\frac{1}{4}x^4+2\frac{1}{3}$.

4. $(x+2)(x+3)$. **5.** $(x^2-x)(2x+1)$. **6.** $(2x^2+5x+1)(x^3+x+3)$.

7. $(x+1)(x+2)(x+3)$. **8.** $(2x-1)(x^3+1)$. **9.** $\dfrac{x+1}{x+2}$.

10. $\dfrac{2x-1}{x^2}$. **11.** $\dfrac{3x+4}{5x-3}$. **12.** $\dfrac{x^2}{x^2+1}$.

13. $\dfrac{(x+1)^2}{x-1}$. **14.** $\dfrac{2x}{(x+1)^2}$. **15.** $\dfrac{x^2-2x-5}{3x-2}$.

16. $(2x+3)^5$. **17.** $(4-3x)^8$. **18.** $(2-x^2)^6$.

19. $(x-3)(x+2)^2$. **20.** $\dfrac{x}{(x-1)^2}$. **21.** $(ax+b)^2(cx+d)^2$.

22. $\dfrac{(5x-3)^3}{(x+2)^4}$. **23.** $\dfrac{(2x-3)^2}{2x-3x^2}$. **24.** $(x^3+x-5)^2$.

25. $\dfrac{1}{x+1}-\dfrac{1}{x+2}$. **26.** $\dfrac{1}{(x^2+4)^2}$. **27.** $\left(\dfrac{1-x}{1+x}\right)^2$.

Derivative of x^n when n is fractional or negative

VII. If n is a positive fraction, $\dfrac{d}{dx}(x^n)=nx^{n-1}$.

Let $n=\dfrac{p}{q}$ where p, q are positive integers.

Let $y = x^n = x^{\frac{p}{q}}$ and put $x^{\frac{1}{q}} = z$ or $x = z^q$,

$$\therefore\ y = x^{\frac{p}{q}} = z^p.$$

Now $\dfrac{dy}{dz} = \dfrac{dy}{dx} \times \dfrac{dx}{dz}$; $\quad \therefore\ \dfrac{dy}{dx} = \dfrac{\frac{dy}{dz}}{\frac{dx}{dz}} = \dfrac{\frac{d}{dz}(z^p)}{\frac{d}{dz}(z^q)}.$

$$\therefore\ \frac{dy}{dx} = \frac{pz^{p-1}}{qz^{q-1}} = \frac{p}{q}\, z^{p-q} = \frac{p}{q}\, x^{\frac{p-q}{q}} = \frac{p}{q}\, x^{\frac{p}{q}-1},$$

$$\therefore\ \frac{d}{dx}(x^n) = \frac{dy}{dx} = \frac{p}{q}\, x^{\frac{p}{q}-1} = nx^{n-1}.$$

VIII. If n is any negative number, $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Let $n = -m$ where m is a positive number.

$$\therefore\ \frac{d}{dx}(x^n) = \frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \frac{d}{dx}(1) - \frac{d}{dx}(x^m)}{x^{2m}} = \frac{0 - mx^{m-1}}{x^{2m}} \text{ by (VI.)}$$

$$= -mx^{m-1-2m} = -mx^{-m-1}$$

$$= nx^{n-1}.$$

We have now proved that the formula

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

holds for all values of n, positive or negative, fractional or integral.

Example 7.

Find $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$ and $\dfrac{d}{dx}(\sqrt{x})$.

$$\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}.$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{\frac{1}{2}-1} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}.$$

Example 8.

Find $\dfrac{d}{dx}(\sqrt{4-x^2})$.

Put $y=\sqrt{4-x^2}$ and $4-x^2=z$, so that $y=\sqrt{z}$.

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{dz}\times\frac{dz}{dx}\\ &= \frac{d}{dz}(z^{\frac{1}{2}})\times\frac{d}{dx}(4-x^2)\\ &= \tfrac{1}{2}z^{-\frac{1}{2}}(-2x)\\ &= -\frac{x}{z^{\frac{1}{2}}} = -\frac{x}{\sqrt{4-x^2}}.\end{aligned}$$

After a little practice it will be found unnecessary to make an actual substitution: the working of this example will then read as follows:

$$\begin{aligned}\frac{d}{dx}(\sqrt{4-x^2}) &= \frac{d\sqrt{4-x^2}}{d(4-x^2)}\times\frac{d(4-x^2)}{dx}\\ &= \tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}\times(-2x)\\ &= -\frac{x}{\sqrt{4-x^2}},\end{aligned}$$

and eventually the first step will be done mentally without actually being written down at all.

EXAMPLES XI b

Differentiate with respect to x the expressions in Examples 1—18:

1. $\dfrac{1}{x^3}$, $x^3-\dfrac{2}{x^2}$, $\dfrac{4}{x}+\dfrac{5}{x^3}$, $\dfrac{10}{x^{10}}$, $\dfrac{1}{x^{2n}}$.

2. $x^{1\cdot5}$, $x^{\frac{2}{3}}$, $\sqrt[3]{x}$, $3x\sqrt{x}$, $\frac{1}{2}\sqrt{x^5}$, $x^{\frac{1}{q}}$, $x^{\frac{2}{n}}$.

3. $\dfrac{1}{\sqrt{x}}$, x^{-3}, $x^{-\frac{1}{3}}$, x^0, $\dfrac{4}{\sqrt[3]{x}}$, $\dfrac{1}{x^{\frac{1}{q}}}$.

4. $x^{-2\cdot3}$, $3x^{-2}$, $\sqrt{\dfrac{5}{x^3}}$, $3x^{\frac{1}{3}}$, $4x^{-\frac{1}{2}}$, $\dfrac{2}{x\sqrt{x}}$, $\sqrt{(5-2x)}$.

5. $\left(x+\dfrac{2}{x}\right)\left(x^2-\dfrac{3}{x^2}\right)$.

6. $(x^{2n}-3)(x^n+1)$.

7. $\dfrac{\sqrt{x}}{1+\sqrt{x}}$.

8. $\dfrac{1}{\sqrt{(3-x)}}$.

9. $\sqrt{(x^2-2x)}$.

10. $\dfrac{1}{\sqrt{(9-x^2)}}$.

11. $\sqrt[3]{(3x^3+8)}$.

12. $\dfrac{x}{\sqrt{(x^2-1)}}$.

13. $x^2\sqrt{(1-x^2)}$.

14. $\sqrt{\left(\dfrac{1+2x}{1-x}\right)}$. **15.** $x+\sqrt{(1+x^2)}$. **16.** $\{x+\sqrt{(1+x^2)}\}^3$.

17. $\dfrac{\sqrt{(2x^2-x-5)}}{1+x}$. **18.** $\dfrac{1}{\sqrt{(x+1)}-\sqrt{x}}$.

19. If $xy^2=4$ express $\dfrac{dy}{dx}$ in terms of x, and $\dfrac{dx}{dy}$ in terms of x. What is $\dfrac{dy}{dx}\times\dfrac{dx}{dy}$?

20. If $y=\sqrt{a^2-x^2}$ where a is a constant, prove that $y\dfrac{dy}{dx}+x=0$.

21. If $y=\dfrac{x}{x+c}$ where c is a constant, prove that $x\dfrac{dy}{dx}=y(1-y)$.

22. If xy is constant, prove that $\dfrac{dy}{dx}=-\dfrac{y}{x}$.

23. Show that $\dfrac{d}{dx}(y^2)=2y\dfrac{dy}{dx}$.

24. If $v=\dfrac{ds}{dt}$, show that $\dfrac{dv}{dt}=\dfrac{d}{ds}(\tfrac{1}{2}v^2)$.

25. If $xy=c^2$ where c is constant and if u is any function of x, show that $x\dfrac{du}{dx}+y\dfrac{du}{dy}=0$.

26. A pendulum of length l feet makes a complete vibration in t secs. where $t=1{\cdot}11\sqrt{l}$. If the length is measured as 4·5 feet, and if this is correct to 2 significant figures, find approximately the possible error in the time when calculated from this formula.

27. Find the stationary values of $\dfrac{(1-x)^3}{1-2x}$.

28. A large stone is dropped into a smooth sheet of water at A; the wave-front reaches a point P, y feet from A, after t seconds where $y=\dfrac{50t}{t+10}$. What is the speed of the wave-front after (i) 1 second, (ii) 10 seconds?

29. For what values of x has the function $(x-2)^{\frac{2}{3}}(x-3)^{\frac{3}{5}}$ a maximum or minimum value?

30. A rectangle is inscribed in a semicircle of radius a so that one side lies on the diameter and the other two corners on the circular arc. Find its maximum area.

31. A, B are two fires 30 feet apart: the fire at A gives out 8 times as much heat as the fire at B. If the intensity of the heat varies inversely as the square of the distance, find what point on AB receives least heat.

32. A light is suspended at a point O, x feet above a table whose top AB is horizontal; A is vertically below O. The intensity of illumination at a point B on the table is proportional to $\frac{OA}{OB^3}$. If $AB=3$ feet, find the height OA in order that the table at B may be illuminated as much as possible.

33. A man is in a boat 2 miles from the nearest point A of a straight sea-coast and wishes to reach a point B on the coast 5 miles from A in the shortest time. He can row 3 miles an hour and walk 4 miles an hour. How far from A should he land?

34. If $y=\frac{x+c}{1+x^2}$ where c is a constant, prove that when y is stationary $2xy=1$.

Implicit Functions

The next example shows how $\frac{dy}{dx}$ may be obtained when x and y are connected by an implicit relation.

Example 9.

To find $\frac{dy}{dx}$, given that $x^2+3xy+y^2=4$.

(i) *From first principles*, if $(x+\delta x, y+\delta y)$ be the coordinates of a point on the curve near (x, y) we have

$$(x+\delta x)^2+3(x+\delta x)(y+\delta y)+(y+\delta y)^2=4,$$

$$\therefore x^2+2x.\delta x+(\delta x)^2+3xy+3x.\delta y+3y.\delta x+3\delta x.\delta y+y^2+2y.\delta y+(\delta y)^2=4.$$

But $x^2+3xy+y^2=4$,

$$\therefore 2x.\delta x+3x.\delta y+3y.\delta x+2y.\delta y+(\delta x)^2+3\delta x.\delta y+(\delta y)^2=0,$$

$$\therefore 2x+3x.\frac{\delta y}{\delta x}+3y+2y.\frac{\delta y}{\delta x}+\delta x+3\frac{\delta y}{\delta x}.\delta x+\left(\frac{\delta y}{\delta x}\right)^2.\delta x=0.$$

$\therefore$ in the limit, when $\delta x \to 0$, we have

$$2x+3x\frac{dy}{dx}+3y+2y\frac{dy}{dx}=0,$$

$$.\ (3x+2y)\frac{dy}{dx}=-(2x+3y),$$

$$\therefore \frac{dy}{dx}=-\frac{2x+3y}{3x+2y}.$$

(ii) *Differentiating by rule* with respect to x.

$$\frac{d}{dx}(x^2)+3\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=0,$$

$$\therefore\ 2x+3\left(y+x\frac{dy}{dx}\right)+\frac{d}{dy}(y^2)\times\frac{dy}{dx}=0,$$

$$\therefore\ 2x+3y+3x\frac{dy}{dx}+2y\frac{dy}{dx}=0,$$

$$\therefore\ \text{as before } \frac{dy}{dx}=-\frac{2x+3y}{3x+2y}.$$

Parametric Notation

x and y are sometimes expressed each in terms of a third variable, called a *parameter.*

Example 10.

To find $\frac{dy}{dx}$, given that $y=3t^2+t$, $x=5t^3-3$.

Since $\frac{dy}{dx}\times\frac{dx}{dt}=\frac{dy}{dt}$, we have $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Now $\frac{dy}{dt}=\frac{d}{dt}(3t^2+t)=6t+1$ and $\frac{dx}{dt}=\frac{d}{dt}(5t^3-3)=15t^2$.

$$\therefore\ \frac{dy}{dx}=\frac{6t+1}{15t^2}.$$

Example 11.

A bird A is flying horizontally 150 feet above the ground at a speed of 10 ft. per sec. Find the rate at which its distance from C is increasing when $CB=200$ feet where B is the point on the ground vertically below A.

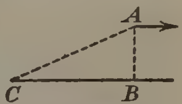

Fig. 128.

AB is constant and $=150$ feet: CB is a variable increasing at the rate of 10 ft. per sec.; let $CB=x$ feet so that $\frac{dx}{dt}=10$ where t is the time measured in seconds.

Let $AC=y$ feet: it is required to find $\frac{dy}{dt}$.

Now $y^2=x^2+150^2$ since $\angle ABC=90°$.

(i) From first principles with the usual notation

$$(y+\delta y)^2=(x+\delta x)^2+150^2.$$

$$\therefore\ y^2+2y\,.\,\delta y+(\delta y)^2=x^2+2x\,.\,\delta x+(\delta x)^2+150^2,$$

but

$$y^2=x^2+150^2,$$

$$\therefore\ 2y\,.\,\delta y+(\delta y)^2=2x\,.\,\delta x+(\delta x)^2.$$

Divide both sides by δt,

$$\therefore\ 2y\frac{\delta y}{\delta t}+\left(\frac{\delta y}{\delta t}\right)^2.\,\delta t=2x\frac{\delta x}{\delta t}+\left(\frac{\delta x}{\delta t}\right)^2.\,\delta t.$$

$\therefore$ in the limit when $\delta t \to 0$ we have

$$2y\frac{dy}{dt}=2x\frac{dx}{dt}.$$

Now when $x=200$, $y^2=200^2+150^2=40{,}000+22{,}500=62{,}500$,

$$\therefore\ y=250,$$

$$\therefore\ \frac{dy}{dt}=\frac{200}{250}\times 10=\underline{8\text{ ft. per sec.}}$$

(ii) By differentiation, $\dfrac{d}{dt}(y^2)=\dfrac{d}{dt}(x^2)+0.$

But $\dfrac{d}{dt}(y^2)=\dfrac{d}{dy}(y^2)\times\dfrac{dy}{dt}=2y\dfrac{dy}{dt}$ and $\dfrac{d}{dt}(x^2)=\dfrac{d}{dx}(x^2)\times\dfrac{dx}{dt}=2x\dfrac{dx}{dt}.$

$$\therefore\ 2y\frac{dy}{dt}=2x\frac{dx}{dt}\text{ as before.}$$

EXAMPLES XI c

Find $\dfrac{dy}{dx}$ for the relations given in nos. 1—12:

1. $x^2+y^2=4$. **2.** $2x^2-3y^2=12$. **3.** $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}$.

4. $x^3+y^3=3xy$. **5.** $xy(x+y)=c$. **6.** $x^3y^2=c$.

7. $xy+2x-3y=1$. **8.** $\sqrt{x}+\sqrt{y}=\sqrt{c}$.

9. $x=at^2,\ y=2at$. **10.** $x=t^2-1,\ y=(t-1)^2$.

11. $x=\dfrac{3m}{1+m^3},\ y=\dfrac{3m^2}{1+m^3}$. **12.** $x=\dfrac{2t}{1+t^2},\ y=\dfrac{1-t^2}{1+t^2}$.

13. The ends P, Q of a rod PQ of length 10 ins. move on two perpendicular lines OA, OB: if P moves steadily at 3 ins. per sec., find the velocity of Q when Q is 6 ins. from O.

14. A stone is thrown so that after t seconds it has travelled $20t$ feet horizontally and $50t-16t^2$ feet vertically. In what direction is it moving after 1 second?

15. A man standing on a wharf is drawing in the painter of a boat at the rate of 2 ft. per sec. His hands are 6 feet above the level of the bow of the boat. How fast is the boat moving through the water when there are still 10 feet of painter out?

16. If $(x-c)(y-c)=1+c^2$ where c is a constant, prove that

$$\frac{dy}{dx}+\frac{1+y^2}{1+x^2}=0.$$

17. If $s=xy$, $t=x+y$, $x^2+y^2=c^2$ where c is a constant, find $\frac{ds}{dt}$ in terms of x, y.

18. If $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, find $\frac{dy}{dx}$.

19. If $xy^2=x^2+4y^2$, find the least positive value of y.

20. A ring is slipping down the rough curve $xy=24$ (unit on each axis 1 foot, y-axis vertical) so that its horizontal velocity is 2 ft. per sec.; what is its vertical velocity when at a height of 6 feet above Ox?

21. One end A of the bounding diameter AB of a semicircular plate moves uniformly at 3 ins. per minute in a slot Oy and the curved rim rolls along a fixed perpendicular line Ox; if $AB=20$ ins., find the rate at which its point of contact P with Ox is moving when $OA=2$ ins.

22. A particle moves s feet in t seconds where $s^3=kt^2$, k being a constant; prove that its acceleration varies as $\frac{1}{s^2}$.

23. A ladder PQ 20 feet long rests with one end P on the ground AP and with the other projecting over a vertical wall AB 12 feet high. The end P is pushed along the ground towards the wall at 2 ft. per sec. What is the *vertical* velocity of Q when P is 5 feet from the wall?

24. A body travels s feet in t secs. where $t=a+bs+cs^2$, a, b, c being constants; prove that its acceleration varies as the cube of its velocity.

25. A rod OP of length l feet is hinged at a point O on the ground and rests against a cylinder of radius a feet which rolls along the ground towards O; if the point of contact Q of the cylinder with the ground is x feet from O when the height of P above the ground is z feet, prove that $z=\frac{2alx}{a^2+x^2}$. If the cylinder is advancing towards O at the rate of a ft. per sec., find the rate at which the height of P is increasing when $OQ=3a$.

26. (i) If $s=at^3$, express, in terms of t, $\frac{ds}{dt}\times\frac{dt}{ds}$; $\frac{d^2s}{dt^2}\times\frac{d^2t}{ds^2}$.

(ii) A body moves s feet in t seconds in such a way that its acceleration varies as the cube of its velocity; prove that $\frac{d^2t}{ds^2}$ is constant.

27. Equal weights of 5 lbs. each are fastened to the ends of a string 30 ins. long which passes over two small smooth pegs A, B in a horizontal line 8 ins. apart. A weight of 6 lbs. is fastened to the middle point of the string. In equilibrium the centre of gravity of the system is at a maximum depth below AB. If the 5 lb. weights are each at a depth of x ins. and the 6 lb. weight at a depth of y ins. below AB, then the depth of the centre of gravity is $\frac{10x+6y}{16}$ ins. Assuming this result, find y in the position of equilibrium.

If the 6 lb. weight is pulled down and released so that it passes through its equilibrium position with a velocity of 8 ins. per sec., what velocity has each 5 lb. weight at this moment?

MISCELLANEOUS EXAMPLES 13—17

M. 13

1. Prove that the sum of the intercepts on the axes made by the tangents to $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ is constant and equal to a.

2. A ship A steams due E. from a port P at 20 knots, setting out at the same moment that another ship B which is steaming at 15 knots towards P is 40 nautical miles due S. of P. Find the distance between the ships at time t hours after A has left P and find for what value of t that distance is least.

3. If the distance s travelled in time t is given by $s=\frac{a}{t}+bt^2$, prove that the acceleration equals $\frac{2s}{t^2}$.

4. ABC is a triangular sheet of thin paper which is being rolled up round AB, a circular rod of radius 1 in., at the rate of 3 turns per sec. Find the rate at which the area ABC is diminishing at the end of the 1st sec., neglecting the increase of radius of the roller due to the rolled up paper.

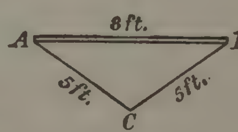

Fig. 129.

5. The sum of the perimeters of two equal squares and a circle is 100 feet. If x is the radius of the circle, draw a graph of A, the total area of the squares and the circle, in terms of x.

When is A least and when greatest? Discuss whether these are maximum and minimum values.

M. 14

1. If O is the origin and P the point (x_1, y_1) on the curve $x^2-y^2=4$, prove that the tangent at P makes the same angle with OX that OP makes with OY.

2. A triangular field ABC has a right angle at A and is divided into two equal parts by a fence PQ of length y cutting AB at P and AC at Q. If $AP=x$, $AB=c$, $AC=b$, prove $y^2=x^2+\dfrac{b^2c^2}{4x^2}$, and find the length of the shortest fence.

3. Sketch the curve $y^2x=(x+1)^2$. Find the abscissa of the point where the tangent is parallel to OX. Show that for values of x greater than this the angle the tangent makes with OX does not exceed the value $\tan^{-1}\dfrac{1}{3\sqrt{3}}$ which occurs when $x=3$. (Cambridge University.)

4. A stone is dropped into a pond and the radius of the outer circular ripple increases steadily at the rate of 5 ft./sec. How rapidly is the area of the disturbed water increasing at the end of 3 secs.?

5. On AB, 6 ins. long, a semicircle is described. C is the mid-point of AB and another semicircle is described on the other side of AB with AC as its diameter. From a point Q on the smaller semicircle QN is drawn perpendicular to AB and produced to meet the other curve in P. Find the maximum value of PQ.

M. 15

1. Find the equations of the tangents to $y=\dfrac{x(x-1)(x+1)}{(x+3)(x-4)}$ where $x=0, +1, -1$. Sketch the curve.

2. A range-finder has a base of d feet which subtends an angle of x seconds at an object whose range is R yards. Show that $x \simeq \dfrac{216{,}000\,d}{\pi R}$. Prove that the error in the range due to a constant small error δx in the angle is proportional to R^2.

3. Two sources of heat at A and B with intensities a and b respectively produce a total intensity at a point P on AB distant x from A given by $I=\frac{a}{x^2}+\frac{b}{(d-x)^2}$ where $AB=d$. Show that the temperature at P will be least when $\frac{d-x}{x}=\sqrt[3]{\frac{b}{a}}$.

4. If $pv^\gamma=c$ for a gas, and h the rate at which it absorbs heat is given by $h=\frac{1}{\lambda-1}\left\{v\frac{dp}{dv}+\lambda p\right\}$, find h in terms of p, λ, γ, and if h is always zero prove $\lambda=\gamma$.

5. The force exerted by a circular electric current of radius a on a small magnet whose axis coincides with the axis of the circle is proportional to $\frac{x}{(a^2+x^2)^{\frac{5}{2}}}$ where x is the distance of the magnet from the plane of the circle. Prove that the force is a maximum when $x=\frac{a}{2}$.

M. 16

1. The weight of gas which will flow per second through an orifice from a vessel where it is at pressure p_1 into another vessel where it is at pressure p_2 is proportional to $a^{\frac{1}{\gamma}}\sqrt{1-a^{\frac{\gamma-1}{\gamma}}}$ where $a=\frac{p_2}{p_1}$ and $\gamma=1{\cdot}41$. Prove that a maximum quantity will leave a vessel per second when the outside pressure is a little greater than half the inside pressure.

2. If $p=\frac{\theta}{v-a}-\frac{c}{\theta(v+b)^2}$, where a, b, c, θ are constants, is plotted as a (p, v) curve, v-axis horizontal, prove that for a certain value of θ there is a horizontal inflection at $v=3a+2b$.

3. The distances u and v of an object and its image from a lens whose focal length is f are related by $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$. If the magnification is $\frac{\text{size of image}}{\text{size of object}}$, prove that the magnification along the axis of a small object of width δu is $\left(\frac{v}{u}\right)^2$ approximately.

4. If x is the number of gram molecules of a substance A which are transformed at a time t by reaction with x gram molecules of another substance B, then $\frac{x}{a-x}=akt$, where a is the number of gram molecules of each of the substances present at the beginning. Show that the velocity of the reaction varies as the square of the amounts of A and B present at time t.

5. When compressed air escapes the rate of decrease of pressure P is proportional to the square root of the difference between P and the atmospheric pressure P_0. If $p=P-P_0$, prove $\frac{dt}{dp}=-\frac{1}{k\sqrt{p}}$. If P reduces from 5 atmospheres to 4 in 1 min., find P after 2 mins.

M. 17

1. If a chord of a circle moves across a circle at right angles to a diameter, find the rate of increase of the area of the segment per unit increase of the chord's distance from the end of the diameter.

2. Find the gradient of the tangent to $y=x^2\sqrt{2+x}$ and state its value where the curve meets OX. Find the position of the maximum ordinate and the angle the tangent makes with OX at the point $x=-1$.

3. Water is escaping from the bottom of a cylindrical vessel at a rate $k\sqrt{x}$ where x is the depth of water remaining. If the area of the water surface is A, and if the initial depth is C, find the time it takes to empty.

4. Prove that $f(x+h) \doteq f(x)+hf'(x)$. One root of $3x^2-x-1=0$ is approximately 0·77. Suppose a more correct value to be $(0{\cdot}77+h)$ and, by using the above relation, find h and hence a better approximation to the root.

5. A body moves so that its velocity after travelling s feet is v ft.-sec. where $v^2=a^2-s^2$; find its acceleration in terms of s.

CHAPTER XII

DERIVATIVES OF THE TRIGONOMETRICAL FUNCTIONS

It is essential that the student should be familiar with the Radian method of measuring angles and with the relation between radians and degrees, before attempting to differentiate the Trigonometrical functions.

Derivative of $\sin x$

Draw a rough graph of $y = \sin x$ from $x = 0$ to $x = 2\pi$ radians. Let (x, y) be the coordinates of P, then $y\,(NP)$ represents the

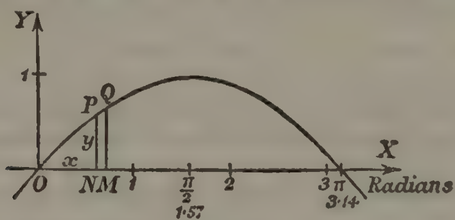

Fig. 130.

sine of the angle whose radian measure is $x\,(ON)$, i.e. $y = \sin x$, and since MQ is the sine of the angle whose radian measure is OM we have

$$y + \delta y = \sin(x + \delta x),$$

$$\therefore\ \delta y = \sin(x + \delta x) - \sin x$$

$$= 2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right).$$

$$\therefore\ \frac{\delta y}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}.$$

Now as $\delta x \to 0$, the limit of

$$\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

equals the limit to which $\frac{\sin\theta}{\theta}$ tends *where θ is measured in radians* if $\theta \to 0$.

This limit was shown to be 1 in Part I, p. 20, hence

$$\frac{d}{dx}(\sin x) = \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \left\{\cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\right\}$$

$$= \cos x.$$

In particular we notice that if Fig. 130 represents the graph of $y = \sin x$, with the same units for OX, OY, then for $x = 0$, the gradient $= \cos 0 = 1$, and so the slope is $\frac{\pi}{4}$; this means that $\sin x$ increases at the same rate as x (in radians) at the origin: also at $x = \frac{\pi}{2}$, the gradient $= \cos\frac{\pi}{2} = 0$ and we have a maximum.

Note. It is most important to realise that the relation

$$\frac{d}{dx}(\sin x) = \cos x$$

is true, *only if* x is measured in radians.

Consider $\frac{d}{dz}(\sin z^\circ)$ where z° means z degrees.

Now $z^\circ = \frac{\pi z}{180}$ radians $= t$ radians, say.

$$\therefore\ \frac{d}{dz}(\sin z^\circ) = \frac{d}{dz}(\sin t) = \frac{d(\sin t)}{dt} \times \frac{dt}{dz}.$$

But $\dfrac{d}{dt}(\sin t) = \cos t$ and $\dfrac{dt}{dz} = \dfrac{\pi}{180}$.

$$\therefore\ \frac{d}{dz}(\sin z^\circ) = \cos t \times \frac{\pi}{180} = \frac{\pi}{180}\cos z^\circ.$$

By employing radians instead of degrees, when differentiating, we avoid this awkward numerical factor.

Derivative of $\cos x$

Fig. 131 represents the graph of $y = \cos x$ between 0 and $\dfrac{\pi}{2}$, x being measured in radians. Since $\cos x$ decreases as x increases it is clear that $\dfrac{d}{dx}(\cos x)$ is negative, if x is acute.

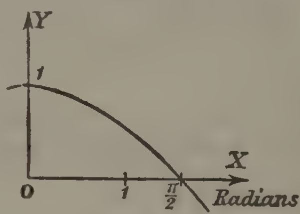

Fig. 131.

As before,

$$\delta y = \cos(x + \delta x) - \cos x$$

$$= -2\sin\left(x + \frac{\delta x}{2}\right) . \sin\frac{\delta x}{2}.$$

$$\therefore\ \frac{\delta y}{\delta x} = -2\sin\left(x + \frac{\delta x}{2}\right) . \frac{\sin\frac{\delta x}{2}}{\delta x} = -\sin\left(x + \frac{\delta x}{2}\right) . \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

$$\therefore\ \frac{d}{dx}(\cos x) = \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}}\left(\frac{\delta y}{\delta x}\right) = -\sin x.$$

Note. (i) We can also obtain this result as follows:

$$\frac{d}{dx}[\cos x] = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \frac{d\sin\left(\frac{\pi}{2} - x\right)}{d\left(\frac{\pi}{2} - x\right)} \times \frac{d\left(\frac{\pi}{2} - x\right)}{dx}$$

$$= \cos\left(\frac{\pi}{2} - x\right) \times (-1) = -\sin x.$$

(ii) It is important to observe as before that the relation

$$\frac{d}{dx}(\cos x) = -\sin x$$

is true *only if* x is measured in radians.

By the same method as above we could show that

$$\frac{d}{dz}(\cos z^\circ) = -\frac{\pi}{180}\sin z^\circ.$$

In future we shall assume, unless otherwise stated, that all angles are measured in radians.

Derivatives of $\sin(ax+b)$ **and** $\cos(ax+b)$

(i) Let $y=\sin(ax+b)$ and $ax+b=u$,

$$\therefore\ y = \sin u,$$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \frac{d}{du}(\sin u)\times\frac{d}{dx}(ax+b)$$

$$= \cos u \times a = a\cos(ax+b).$$

(ii) Similarly

$$\frac{d}{dx}[\cos(ax+b)] = \frac{d\cos(ax+b)}{d(ax+b)}\times\frac{d(ax+b)}{dx}$$

$$= -\sin(ax+b)\times a = -a\sin(ax+b).$$

Alternative Geometrical Method

The derivatives of $\sin\theta$ and $\cos\theta$ may also be obtained as follows:

Draw a circle, centre the origin O, radius unity.

Let $\angle NOP = \theta$ radians, $\angle POQ = \delta\theta$ radians. R is the foot of the perpendicular from P to QM.

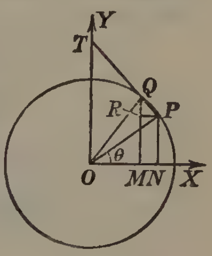

Fig. 132.

Then $\qquad y = NP = \sin\theta,$

$$y + \delta y = MQ = \sin(\theta + \delta\theta).$$

$$\therefore\ \frac{d}{d\theta}(\sin\theta) = \frac{dy}{d\theta} = \underset{\delta\theta\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta\theta}$$

$$= \mathrm{Lt}\,\frac{MQ - NP}{\text{arc } PQ} = \mathrm{Lt}\,\frac{RQ}{\text{arc } PQ} = \mathrm{Lt}\,\frac{RQ}{\text{chord } \overline{PQ}}\times\frac{\text{chord } \overline{PQ}}{\text{arc } PQ}.$$

Now, if PQ cuts OY at T,

$$\frac{RQ}{PQ} = \cos RQP = \cos OTP.$$

But when $\delta\theta \to 0$, TP becomes the tangent at P and

$$\angle OTP = \frac{\pi}{2} - \angle TOP = \theta,$$

$$\therefore \ \text{Lt} \frac{RQ}{PQ} = \cos\theta.$$

Also $\text{Lt} \dfrac{\text{chord } PQ}{\text{arc } PQ} = 1$,

$$\therefore \ \frac{d}{d\theta}(\sin\theta) = \cos\theta,$$

and similarly

$$\frac{d}{d\theta}(\cos\theta) = \frac{dx}{d\theta} = \underset{\delta\theta \to 0}{\text{Lt}} \frac{\delta x}{\delta\theta} = \text{Lt} \frac{OM - ON}{\text{arc } PQ} = -\text{Lt} \frac{MN}{\text{arc } PQ}$$

$$= -\text{Lt} \frac{RP}{\text{chord } PQ} \times \frac{\text{chord } PQ}{\text{arc } PQ} \qquad \text{since } MN = RP,$$

but $\quad \dfrac{RP}{PQ} = \sin RQP = \sin OTP \to \sin\theta \qquad$ when $\delta\theta \to 0$,

$$\therefore \ \frac{d}{d\theta}(\cos\theta) = -\sin\theta.$$

Derivative of $\tan x$

If $y = \tan x, \qquad y + \delta y = \tan(x + \delta x)$,

$$\therefore \ \delta y = \tan(x + \delta x) - \tan x$$

$$= \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x) \,.\, \cos x}$$

$$= \frac{\sin\{(x+\delta x) - x\}}{\cos(x+\delta x) \,.\, \cos x} \qquad \text{cf. } \sin(A - B)$$

$$= \frac{\sin \delta x}{\cos(x+\delta x) \,.\, \cos x}.$$

$$\therefore \ \frac{\delta y}{\delta x} = \frac{1}{\cos(x+\delta x) \,.\, \cos x} \,.\, \frac{\sin\delta x}{\delta x},$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x.$$

This result shows that the gradient of $\tan x$ is always positive, as is clear from a graph of the function.

Alternative Method.

$$y = \tan x = \frac{\sin x}{\cos x}$$

may be differentiated by using the rule for a quotient,

$$\therefore \frac{dy}{dx} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Derivative of $\operatorname{cosec} x$

Proceeding as before we have

$$\begin{aligned} \delta y &= \operatorname{cosec}(x + \delta x) - \operatorname{cosec} x \\ &= \frac{1}{\sin(x + \delta x)} - \frac{1}{\sin x} \\ &= \frac{\sin x - \sin(x + \delta x)}{\sin(x + \delta x) \,.\, \sin x} \\ &= -\frac{2\cos\left(x + \frac{\delta x}{2}\right) . \sin\frac{\delta x}{2}}{\sin(x + \delta x) \,.\, \sin x}. \end{aligned}$$

$$\therefore \frac{\delta y}{\delta x} = -\frac{\cos\left(x + \frac{\delta x}{2}\right)}{\sin(x + \delta x) \,.\, \sin x} \cdot \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}},$$

$$\therefore \frac{dy}{dx} = -\frac{\cos x}{\sin x \,.\, \sin x} = \underline{-\cot x \,.\, \operatorname{cosec} x.}$$

Here again the derivative is negative since $\operatorname{cosec} x$ decreases when x increases from 0 to $\frac{\pi}{2}$.

Alternative Method.

$$y = \operatorname{cosec} x = (\sin x)^{-1}.$$

Let $u = \sin x$, then $y = u^{-1}$ and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times \cos x = -\frac{\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x.$$

Derivatives of $\sec x$ **and** $\cot x$

The method is similar to that used for the other functions and the details are left to the student: we find

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

Summary.

y	$\frac{dy}{dx}$ (x in radians)	$\frac{dy}{dx}$ (x in degrees)
$\sin x$	$\cos x$	$\frac{\pi}{180}\cos x$
$\cos x$	$-\sin x$	$-\frac{\pi}{180}\sin x$
$\tan x$	$\sec^2 x$	$\frac{\pi}{180}\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$-\frac{\pi}{180}\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$	$\frac{\pi}{180}\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$	$-\frac{\pi}{180}\operatorname{cosec}^2 x$

The results given in this table include the cases where the angle is given in degrees in order to emphasise the differences that occur when degrees are used instead of radians. But the student should always work in radian measure (and so use the simpler formulae) whenever possible.

Example 1.

Differentiate $\sin(2x+4)$.

$$\begin{aligned}\frac{d}{dx}\sin(2x+4) &= \frac{d\sin(2x+4)}{d(2x+4)} \times \frac{d(2x+4)}{dx}\\ &= \cos(2x+4)\times 2\\ &= 2\cos(2x+4).\end{aligned}$$

Example 2.

Differentiate $\tan^2(3x+2)$.

Let $y=\tan^2(3x+2)$ and put $u=\tan(3x+2)$, then $y=u^2$.

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$=2u\times\sec^2(3x+2)\times 3$$
$$=6\sec^2(3x+2)\tan(3x+2).$$

Or thus:

$$\frac{d}{dx}[\tan^2(3x+2)]=\frac{d[\tan^2(3x+2)]}{d[\tan(3x+2)]}\times\frac{d[\tan(3x+2)]}{d(3x+2)}\times\frac{d(3x+2)}{dx} \quad \text{(done mentally)}$$
$$=2\tan(3x+2)\,.\,\sec^2(3x+2)\,.\,3.$$

EXAMPLES XII a

Differentiate with respect to x the expressions in Ex. 1—16:

1. $\sin(3x+4)$, $\cos 3x$, $\sin(2-x)$.
2. $\tan 4(x-2)$, $\cos(a-x)$, $\sin\frac{2\pi}{3}(x+4)$.
3. $\sec 2x$, $\sec(3-x)$, $\tan(4x-3)$.
4. $\operatorname{cosec} 4x$, $\sin\frac{2\pi}{a}(x-b)$, $\operatorname{cosec}(3x-1)$.
5. $\cot ax$, $\cot(2-x)$, $\tan 3(2-x)$.
6. $\sin^2 x$, $\cos^2 2x$, $\tan^2(x-2)$.
7. $\sin^2(3x-4)$, $\cos^3 3x$, $\operatorname{cosec}^2(x-2)$.
8. $\sec^2(ax-b)$, $\operatorname{cosec}^3 2x$, $\cot(x^{\frac{1}{2}})$.
9. $x\sin x$, $x^2\cos x$, $x\sin x+\cos x$.
10. $\tan x-x$, $x\tan x$, $\cos 2x\sin x$.
11. $\sin x\sin 2x$, $\sin ax\cos bx$, $\cos px\cos qx$.
12. $\frac{1}{2}x+\frac{\sin 2x}{4}$, $\tan 2x\cos x$, $x^2\operatorname{cosec} x$.
13. $\frac{\sin x}{x}$, $\frac{1-\sin x}{1+\sin x}$, $\frac{1}{1+\sin x}$.
14. $\frac{\tan x-1}{\sec x}$, $\cos\frac{a}{x}$, $\sqrt{\sin 2x}$.
15. $\sec^2\frac{x}{2}$, $\tan^2\frac{x}{2}$, $\frac{\cos x}{\sin x+\cos x}$.
16. $\sin(2x^\circ)$, $\cos^2(x^\circ)$, $\tan(\frac{1}{2}x^\circ)$.

Further illustrations

In effecting the inverse process of integration we notice that since

$$\frac{d}{dx}(\sin nx) = n \cos nx,$$

the solution to the equation

$$\frac{dy}{dx} = \cos nx$$

is given by
$$y = \frac{1}{n}\sin nx + c,$$

where c is a constant.

In order to integrate powers of $\sin x$ or $\cos x$ we first express them as functions of multiple angles of the first degree.

Example 3.

If $\dfrac{dy}{dx} = \sin^2 x$, find y.

We have
$$\frac{dy}{dx} = \tfrac{1}{2}(1 - \cos 2x) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x.$$

But
$$\frac{d}{dx}(\sin 2x) = 2\cos 2x,$$

$$\therefore\ y = \frac{x}{2} - \tfrac{1}{4}\sin 2x + c,$$

where c is a constant.

Example 4.

A bird A is flying horizontally 150 ft. above the ground at a speed of 10 ft./sec. Find the rate at which its distance from C is increasing when $CB = 200$ ft. (cf. p. 155).

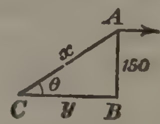

Fig. 133.

Let $A\hat{C}B$ be θ radians, and CA be x.

Then $x = 150 \operatorname{cosec}\theta$,

$$\therefore\ \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$= -150 \operatorname{cosec}\theta \cot\theta \times \frac{d\theta}{dt} \quad \text{.................(i).}$$

If $CB = y$, then $y = 150 \cot\theta$,

$$\therefore\ \frac{dy}{dt} = -150 \operatorname{cosec}^2\theta \,.\, \frac{d\theta}{dt} \quad \text{...................(ii).}$$

But when $y=200$, $\dfrac{dy}{dt}=10$ and $CA=250$,

$$\therefore \operatorname{cosec}\theta=\tfrac{250}{150}=\tfrac{5}{3},$$

$$\therefore 10=-150\times\tfrac{25}{9}\times\frac{d\theta}{dt} \qquad \text{from (ii)},$$

$$\therefore \frac{d\theta}{dt}=-\tfrac{3}{125} \quad \text{.........................(iii)},$$

$$\therefore \frac{dx}{dt}=-150\times\tfrac{5}{3}\times\tfrac{4}{3}\times(-\tfrac{3}{125}) \qquad \text{from (i)}$$

$$=\underline{8 \text{ ft./sec.}}$$

From (iii) we see that at this moment the rate of diminution of θ is $\tfrac{3}{125}$ rad./sec.

Example 5. *Motion of a crank.*

If OP is a crank rotating about O and PQ is the connecting-rod, then the velocity of Q at any time t is $\dfrac{dx}{dt}$.

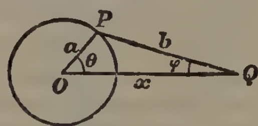

Fig. 134.

But $\qquad x=a\cos\theta+b\cos\phi,$

$$\therefore \frac{dx}{dt}=a\frac{d}{dt}(\cos\theta)+b\frac{d}{dt}(\cos\phi)$$

$$=a\frac{d}{d\theta}(\cos\theta)\times\frac{d\theta}{dt}+b\frac{d}{d\phi}(\cos\phi)\frac{d\phi}{dt},$$

or
$$\dot{x}=-a\sin\theta\,\dot{\theta}-b\sin\phi\,\dot{\phi}.$$

We can thus obtain the velocity of Q in terms of the angular velocities of OP and PQ.

EXAMPLES XII b

1. Find y if (i) $\dfrac{dy}{dx}=\cos 2x$, (ii) $\dfrac{dy}{dx}=\sin 3x$.

2. Find y if (i) $\dfrac{dy}{dx}=\sec^2 2x$, (ii) $\dfrac{dy}{dx}=\tan^2 2x$.

3. Find y if (i) $\dfrac{dy}{dx}=\sin ax\cos bx$, (ii) $\dfrac{dy}{dx}=\cos^2 x$.

4. Prove that $\qquad y=a\cos nx+b\sin nx$

is a solution of the equation $\dfrac{d^2y}{dx^2}+n^2y=0$.

5. Prove that $\qquad y=a\sin(nx+\alpha)$

is a solution of $\qquad \dfrac{d^2y}{dx^2}+n^2y=0.$

6. If $y = \tan x$, prove that
$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}.$$

7. Find the maximum and minimum values of $4\cos x + 3\sin x$.

8. OX and OY are two rods at right angles. A rod AB 2 ft. long moves with one end A on OY and the other end B on OX. If A is descending towards O at the rate of 2 ft./sec. when AB makes an angle of 30° with OX, find the rate at which B is moving.

9. Calculate $\sin(x + 30') - \sin(x - 30')$ for $x = 0, 20°, 40°, 60°$ and show that the values are approximately proportional to the corresponding values of $\cos x$. If the expression equals $k\cos x$, explain why k is approximately the same as $\frac{\pi}{180}$.

10. A point P describes a circle, centre O, radius a, with uniform angular velocity ω, starting from A, the end of the diameter $A'OA$. PN is the perpendicular from P to $A'A$ after a time t. If $ON = x$, find the velocity of N and also its acceleration in terms of ω, a, t.

11. If x is the deflection in a tangent galvanometer and a given small error is made in reading the deflection, show that the *percentage* error in the current C is proportional to $(\tan x + \cot x)$ where $C = k\tan x$.

12. A particle moves in a straight line so that its distance x from a fixed point in the line at time t is given by $x = a\cos(nt + \epsilon)$; prove that its retardation is proportional to x.

13. Show that $\frac{1 + x\tan x}{x}$ is a minimum when $x = \cos x$. From the tables show that $x \triangleq 0{\cdot}739$.

14. Show that the maximum value of $a\sin x + b\cos x$ is $+\sqrt{a^2 + b^2}$ and the minimum value is $-\sqrt{a^2+b^2}$. Also solve this question by writing $a\sin x + b\cos x$ in the form $k\sin(x + \theta)$ where $\theta = \tan^{-1}\frac{b}{a}$.

15. A string, one end of which is fastened at A, passes over a smooth pulley B and carries a weight W at its other end: a smooth ring C (see Fig. 135) slides on the string: AB is horizontal and $= 2a$: $AC = CB = x$, $\angle CBA = \theta$. If W descends with velocity v, find the rate of change of θ and the rate of ascent of C, each in terms of a, v, θ, and evaluate each when $\theta = \frac{\pi}{4}$.

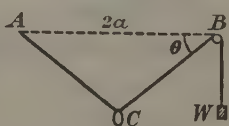

Fig. 135.

16. Fig. 136 represents a rod AB hinged at B and a rod CD hinged to AB at C: the end D slides in the groove BE. If

$$AC = CB = CD = 2 \text{ ft.},$$
$$\angle BCD = \theta^\circ,$$

find the vertical rate of descent of A and the speed of D when $\theta = 90^\circ$, given that θ is increasing at the uniform rate of 1° per sec.

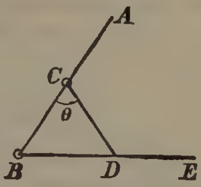

Fig. 136.

Derivatives of Inverse Trigonometrical Functions

If $x = \sin y$, then $y = \sin^{-1} x$ where $\sin^{-1} x$ is written for "the angle whose sine is x."

If we are given $x = \cdot 5$ (say), then y may be $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$ ...; it is therefore a multiple-valued function of x from $x = -1$ to $+1$ both inclusive.

The graph of $y = \sin^{-1} x$ shown in Fig. 137 is obviously the same curve as $x = \sin y$.

If we have the graph of $y = \sin x$ already drawn we can obtain the graph of $y = \sin^{-1} x$ by taking the image of the graph of $y = \sin x$ in the line $y = x$.

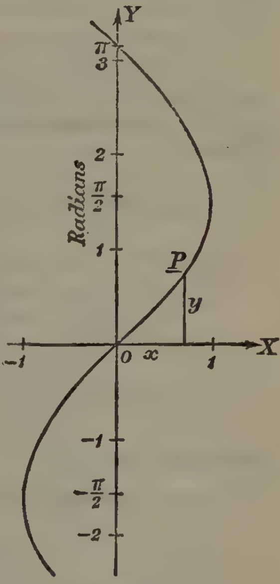

Fig. 137.

Derivative of $\sin^{-1} x$

Let $\qquad y = \sin^{-1} x,$

then $\qquad x = \sin y,$

$$\therefore \frac{dx}{dy} = \cos y,$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\pm\sqrt{1 - \sin^2 y}} = \frac{1}{\pm\sqrt{1 - x^2}}.$$

We see from the graph that the gradient is positive between $y = -\frac{\pi}{2}$ and $+\frac{\pi}{2}$. These values of y are called its *Principal Values*

and the derivatives for these values of x will be given by taking the positive sign of the square root.

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Derivative of $\cos^{-1} x$

Fig. 138 represents the graph of $y = \cos^{-1} x$; then, proceeding as before, we obtain

$$\frac{dy}{dx} = -\frac{1}{\pm\sqrt{1-x^2}}.$$

The principal values of $\cos^{-1} x$ are those between $y = 0$ and π: for these values the gradient of the function is negative and we obtain them by again taking the positive value of the square root.

$$\therefore \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.$$

Note. Since

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},$$

$$\frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x) = 0.$$

$$\therefore \frac{d}{dx}(\cos^{-1} x) = -\frac{d}{dx}(\sin^{-1} x).$$

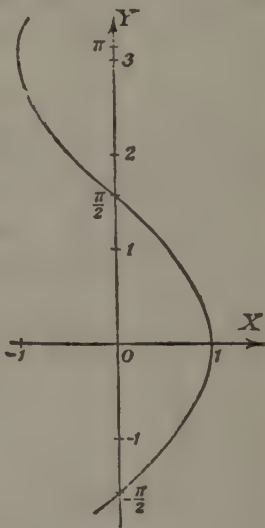

Fig. 138

Derivative of $\tan^{-1} x$

A similar procedure gives us

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Here there is no ambiguity of sign, since the gradient is always positive. The principal values of $\tan^{-1} x$ are those from

$$y = -\frac{\pi}{2} \text{ to } +\frac{\pi}{2}.$$

If $y = \sin^{-1}\dfrac{x}{a}$ we have, by putting $z = \dfrac{x}{a}$,

$$y = \sin^{-1} z$$

and
$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{1}{\sqrt{1-z^2}} \times \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}.$$

Similarly
$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2-x^2}}.$$

But
$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2}.$$

These results are chiefly of importance as a clue to methods of integration, and further reference to them will be found in Chapter XIV.

EXAMPLES XII c

Differentiate with respect to x the expressions in Ex. 1—12:

1. $\sin^{-1}(2x)$. **2.** $\cos^{-1}(3x)$. **3.** $\tan^{-1}(4x)$. **4.** $\sin^{-1}\left(\dfrac{x}{3}\right)$.

5. $\dfrac{1}{x}\sin^{-1}(x)$. **6.** $x^2\cos^{-1}(x)$. **7.** $\operatorname{cosec}^{-1}\left(\dfrac{x}{a}\right)$. **8.** $\sec^{-1}\left(\dfrac{x}{a}\right)$.

9. $\cot^{-1}\left(\dfrac{x}{a}\right)$. **10.** $\tan^{-1}(ax)+\cot^{-1}(ax)$.

11. $(1+x^2)\tan^{-1}x - x$. **12.** $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

13. If $\dfrac{dy}{dx} = \dfrac{1}{a^2+x^2}$ and if $x = a\tan\theta$, find $\dfrac{dy}{d\theta}$.

14. If $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2-x^2}}$ and if $x = a\sin\theta$, find $\dfrac{dy}{d\theta}$.

15. If $\dfrac{dy}{dx} = \dfrac{1}{4+x^2}$, find y.

16. If $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(9-x^2)}}$, find y.

17. If $\dfrac{dy}{dx} = \dfrac{1}{x^2+2x+10}$, find y.

18. Sketch the graph of $y=\cos x$ from $x=0$ to $x=2\pi$; sketch the reflection of this in the line $y=x$. If the equation of this is $y=f(x)$, what is $f(x)$?

REVISION PAPERS 12—17

R. 12

1. If $y=\dfrac{\cos ax}{1+x}$, prove that
$$\frac{d^2y}{dx^2}+\frac{2}{1+x}\frac{dy}{dx}+a^2y=0.$$

2. Find the area of the loop of $y^2=x(x-1)^2$.

3. Differentiate:
$$\text{(i) } \frac{x+1}{\sqrt{x}}; \quad \text{(ii) } \sqrt{(x^n+2)}; \quad \text{(iii) } \frac{\sin x}{1-\cos x}.$$

4. A, B are two points on the same side of the line CD; AH, BK are the perpendiculars from A, B to CD; P is a point on HK; if $AH=1$, $HK=9$, $BK=2$, $HP=x$, find the value of x for which $AP+PB$ is least and prove that in this case $\angle APH=\angle BPK$.

5. (i) Find y if $\dfrac{dy}{dx}=\sin 2x \sin 3x$;

(ii) If $y=\tan^{-1}x$ and $z=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, find $\dfrac{dy}{dz}$.

R. 13

1. If $y=\dfrac{a\cos kx+b\sin kx}{x}$, prove that
$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+k^2y=0.$$

2. A cistern without a lid has a square base: the area of the metal used for its base and sides is 300 sq. ft. What is its maximum volume?

3. Find the position of the centre of gravity of the area bounded by the curve $y^2=x(x-1)^2$ between $x=0$ and $x=1$.

4. Differentiate:
$$\text{(i) } \frac{1}{4-x^3}; \quad \text{(ii) } x-\sqrt{(1+x^2)}; \quad \text{(iii) } \sec^3 x.$$

5. If $xy^3=8$, express $\dfrac{dy}{dx}$ in terms of x and $\dfrac{dx}{dy}$ in terms of x. What is $\dfrac{dx}{dy}\times\dfrac{dy}{dx}$?

R. 14

1. A circular cylinder is closed at one end and open at the other: its total surface is 10 sq. ins. What is its maximum volume?

2. What is the shape of the curve

$$y^2 = x^2(16 - x^2)?$$

It is rotated about the x-axis: what is the volume of the solid so generated?

3. Differentiate:

$$\text{(i)}\ \frac{2-x}{1+x^2}; \quad \text{(ii)}\ \sec^{-1}(2x); \quad \text{(iii)}\ \cos 3x^\circ.$$

4. A point is moving round a vertical circle at the uniform rate of 10 revolutions per minute; its shadow is projected vertically downwards on to the ground; find the speed of its shadow when it is vertically below the mid-point of a horizontal radius, if the radius is 6″.

5. The mixture inside the cylinder of a petrol engine is compressed according to the law $pv^{1 \cdot 4} = \text{constant}$: at the beginning of the stroke $p = 20$, $v = 60$; at the end of the stroke $v = 15$. Find the work done in compressing the mixture.

R. 15

1. If $x = \dfrac{2t}{1+t^2}$, $y = \dfrac{1-t^2}{1+t^2}$, express $\dfrac{dy}{dx}$ in terms of t.

2. A particle can slide along the curve $x^2 + 2y^2 = 3$: it is placed at the point $(1, 1)$: if in a small displacement its x-coordinate is increased by $0 \cdot 1$, what is the approximate change in its y-coordinate?

3. Differentiate: (i) $\sin(\sqrt{x})$; (ii) $\sqrt{(\sin x)}$.

4. Find the area of the segment cut off from the parabola

$$y = x(1-x)$$

by the line $x = 4y$.

5. Draw a square $ABCD$ and a line parallel to DC cutting AD, BC at P, Q; $ABCD$ represents a uniform sheet of metal of side 10″ and $PDCQ$ is a strip of similar metal attached to it. Find the length of PD if the centre of gravity of the composite sheet is at a maximum distance from AB.

R. 16

1. By putting $y=tx$, express the x, y coordinates of any point on the curve $x^3+y^3=3xy$ in terms of t, and find the value of t for which the tangent is parallel (i) to the x-axis, (ii) to the y-axis.

2. Differentiate: (i) $\dfrac{x^n}{\underline{|n}}$; (ii) $\sin^2 x+\cos^2 x$.

If $y=\sin(ax)$, find $\dfrac{d^4y}{dx^4}$.

3. If a wheel of radius a is rolling along the ground and if ω is its angular velocity, the position of a point on the rim after t secs. is given by $x=a[\omega t-\sin(\omega t)]$, $y=a[1-\cos(\omega t)]$, find the velocity of the point at that moment.

4. Fig. 139 represents a T-shaped strip of metal: if $AD=6''$, $AB=2''$, $EF=4''$, $FG=1''$, find the radius of gyration (i) about AB, (ii) about AD.

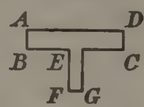

Fig. 139.

5. The power required to drive an aeroplane is measured by $a\sqrt{\theta}+\dfrac{b}{\sqrt{\theta^3}}$, where θ is the angle of attack and a, b are constants; for what value of θ is the power least?

R. 17

1. (i) If $\sin y=\tan x$, express $\dfrac{dy}{dx}$ in terms of x.

(ii) If $y=\cos x$, prove that $\dfrac{d^ny}{dx^n}=\cos\left(x+\dfrac{n\pi}{2}\right)$

2. If the area of a triangle is calculated from the formula $\Delta=\frac{1}{2}bc\sin A$ and if b, c are measured correctly but A is taken as 60° with a possible error of 5′, calculate the possible percentage error in Δ.

3. Evaluate:

$$\text{(i) } \int_0^1 x(1-x)^3\,dx; \quad \text{(ii) } \int_1^2 \frac{dx}{x^{0\cdot 3}}; \quad \text{(iii) } \frac{d}{dx}\sin\left(\frac{1}{x}\right).$$

4. A pressure of 25 lbs. will keep a certain spiral spring compressed through 3 ins.; how much work is needed to compress it an extra inch?

5. A closed vessel is such that when the depth of water in it is x feet, the volume of water is $3\sin^2 x+2\sin 2x$ cu. ft., the angles being in radians. Show that the greatest height of the vessel is about 13·3 ins. and find the area of the cross-section at a height of 3 ins.

MISCELLANEOUS EXAMPLES 18—23

M. 18

1. A small revolving mirror at A reflects a spot of light on to a screen 50 cms. away. If the mirror is rotating at the rate of 10° per second, find the speed of the spot of light when it is 10 cms. from its nearest position to A.

2. P is any point on the arc of a semicircle APB (see Fig. 140); BM is perpendicular to the tangent at P and MR is perpendicular to AB. If $\angle BAP=\theta$, $AB=2a$, find MR and determine for what value of θ it is a maximum.

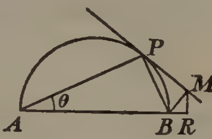

Fig. 140.

3. If the sides a and b of a triangle are measured correctly, but there is an error of δC in the angle C, prove that the error in the area is $\frac{1}{2}ab\cos C \cdot \delta C$. Find the error in the area if $a=325{\cdot}0$, $b=245{\cdot}0$, $C=60°$, $\delta C=6'$.

4. If $V=a\sin(\omega t-nx)$, calculate $\dfrac{d^2V}{dt^2}$ if a, ω, n, x are constants and $\dfrac{d^2V}{dx^2}$ if a, ω, n, t are constants, and find their ratio.

5. In Rapson's steering gear the tiller AB is worked by a carriage CD which runs on rails PQ, $P'Q'$, backwards and forwards athwart the ship, the carriage being pulled by a chain MN which is kept parallel to the rails. The carriage carries a swivelling block EF having a hole in it, into which slides the end of the tiller. B is the axle of the rudder and θ is the angle GBH which the tiller makes with the fore and aft line. If the carriage moves a small distance δx, prove that the angle turned through by the tiller $=\dfrac{\delta x}{a}\cos^2\theta$ radians approximately.

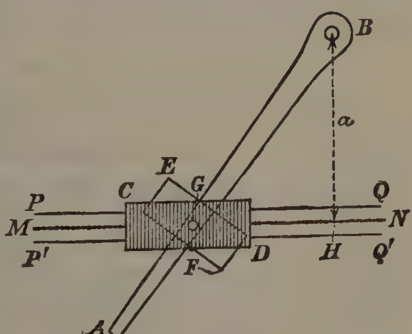

Fig. 141.

(Army.)

M. 19

1. If the distance x ft. a body has travelled in time t secs. is given by $x=1+3\sin 2t-\cos 2t$, find the value of t at which the body first comes to rest and find its acceleration at that time.

2. For a tangent galvanometer $C=\kappa\tan\theta$ where C is the current and θ the deflection. Prove that the proportional error in C due to a given error of reading is least when $\theta=45°$.

3. Find by the Calculus the value of sec 60° 1′, given sec 60° = 2.

4. For an electric circuit

$$V=RC+L\frac{dC}{dt} \text{ and } C=a\sin\kappa t;$$

prove

$$V=\sqrt{(Ra)^2+(La\kappa)^2}\sin(\kappa t+\theta),$$

where

$$\tan\theta=\frac{L\kappa}{R}.$$

5. A rectangular box whose base is a square of side a ft. and whose height is h ft. contains water. It is tilted about BC at a constant angular velocity ω rads./sec. Find the rate at which the wetted surface of AF diminishes up to $\theta=\tan^{-1}\frac{h}{a}$ and the surface of AC diminishes after that value.

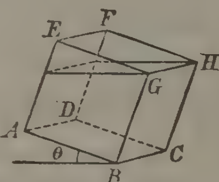

Fig. 142.

M. 20

1. Show that the point $x=a\cos\theta$, $y=b\sin\theta$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Find the gradient of the tangent at this point in terms of θ.

2. A rod AB of length 16 cms. rests between a wall AD and a smooth peg C, 1 cm. from the wall, and makes an angle θ with the horizontal. Prove that the height of G, the mid-point of the rod above the peg, is

$$8\sin\theta-\tan\theta$$

and find for what value of θ this height is a maximum.

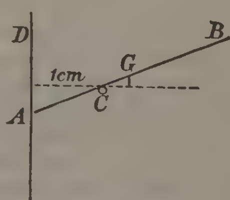

Fig. 143.

3. If $y = uv$, prove that the percentage increase in y due to small increments δu and δv is approximately

$$\left(\frac{\delta u}{u} + \frac{\delta v}{v}\right) 100.$$

The intensity of illumination at a point A of a surface due to a source of light at $P = \kappa \dfrac{\cos\theta}{r^2}$, where $r = AP$ and θ is the angle the normal to the surface at A makes with AP. Prove that the percentage increase of illumination at a point near A where r is increased by δr and θ by $\delta\theta$ radians is

$$-100\left(\tan\theta\,\delta\theta + \frac{2\delta r}{r}\right).$$

Evaluate when $\theta = 45°$, $r = 30$ cms., $\delta\theta = 1°$, $\delta r = 1$ cm. (Army.)

4. P the end of a crank moves uniformly round a circle of radius a, centre O. Q, the other end of the connecting rod PQ, of length b, moves along a straight line through O. If OP makes an angle θ with OQ, when QP makes an angle ϕ with QO prove that

$$\frac{\text{vel. of } Q}{\text{vel. of } P} = \frac{\sin(\theta + \phi)}{\cos\phi}.$$

5. A load of weight W is being drawn along a rough horizontal surface by a rope inclined at an angle θ to the ground. Find by the Calculus the value of θ for which the tension is least, if λ is the angle of friction.

M. 21

1. A point moves so that its position at the end of t secs. is given by

$$x + 2 = \cos\frac{\pi t}{4} + \cos\frac{2\pi t}{4},$$

$$y = \sin\frac{\pi t}{4}.$$

Find its position, velocity and acceleration when $t = 3$: x and y being in feet.

2. The jib of a crane is 20 ft. long and the vertical velocity of its end is 1 ft. per sec. What is the angular velocity of the rotation of the jib when it makes 30° with the vertical?

3. Fig. 144 represents a stage in a total solar eclipse, the Sun and Moon being of equal radius a. The shaded circle, centre O, represents the Moon moving at a uniform rate across the Sun whose centre is at O' and exactly covering it at the moment of totality. Find an expression in terms of θ radians for the fraction of area of the Sun which is uncovered. Calling the uncovered area A, find $\frac{dA}{d\theta}$. Show that $\frac{dA}{dr}$ = length of chord PQ where $OO'=r$. (Army.)

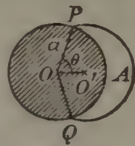

Fig. 144.

4. Prove that as x increases $\frac{a \sin x + b \cos x}{c \sin x + e \cos x}$ either increases for all values of x or else decreases for all values of x: find the condition to be satisfied by the constants a, b, c, e to provide that it shall always increase. (Cambridge University.)

5. A man is walking at 3 mls./hr. down a road 2 ft. from a wall BC. A road EF which turns off at right angles is 24 ft. wide. If A looks past the corner C at a wall EF on the opposite side of the other road, at what rate does this wall open up when AC makes an angle θ with CB? Prove that when $\theta = 45°$ the acceleration of G is 232·32 ft./sec.^{2}

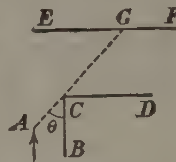

Fig. 145.

M. 22

1. The coordinates of a point are given by

$$x = 1 - \cos\theta, \quad y = \cos\theta + \tfrac{1}{4}\cos 2\theta:$$

prove that

$$\frac{dy}{dx} = -2\cos^2\frac{\theta}{2}.$$

2. A pole 30 ft. long is carried along a passage 12 ft. wide and into a corridor at right angles to the passage. Neglecting the thickness of the pole, find how wide the corridor must be in order that the pole may go round the corner without tilting one edge higher than the other.

3. If $\frac{\sin\phi}{\sin\phi'} = \mu\ (\mu > 1)$, prove

$$\frac{\tan\frac{\phi - \phi'}{2}}{\tan\frac{\phi + \phi'}{2}} = \frac{\mu - 1}{\mu + 1}.$$

Hence show that if ϕ increases then $\phi - \phi'$ also increases.

Also prove $\left(\dfrac{d\phi}{d\phi'}\right)^2 = \dfrac{\mu^2 - 1}{\cos^2\phi} + 1$ and hence show that the increase of ϕ is greater than the increase of ϕ'.

4. A rod CD hinged at C is kept in contact with a door AB by a spring, so that as AB turns round A, D slides along BA. If the angular velocity of the door is ω, find the speed of D along BA when $\angle CDA = \theta$ and $AD = x$.

(Put $AC = d$ and $x = a\cos\theta + d\cos\phi$.)

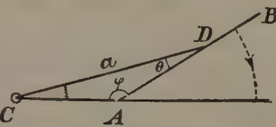

Fig. 146.

5. A rod AB of length $2l$ rests between two planes OP, OQ facing one another, whose inclinations are α and β. G, the centre of gravity of the rod, is at a distance c from A. If AB is inclined at θ to the horizontal, find the height of G above O and find for what value of θ it is a **maximum**.

M. 23

1. The time of oscillation of a pendulum is $t = 2\pi\sqrt{\dfrac{l}{g}}$ and g varies inversely as the square of the distance of the place from the centre of the earth. If the radius of the earth is taken as 4000 miles and a clock with a seconds pendulum ($t=2$) is correct at sea-level, how much will it go wrong per day if carried to a height of $\frac{1}{10}$ of a mile?

2. A vertical wheel is revolving about its centre. If the radius is 4 ft., compare the vertical speed of a point on the rim with its horizontal speed when the abscissa of the point is 2 and the origin is at the centre.

3. A point moves in a circle of radius a with constant angular velocity ω. Show that after time t the coordinates of the point are $x = a\cos\omega t$ and $y = a\sin\omega t$ if $y=0$ when $t=0$.

Prove $$\ddot{x}\cos\omega t + \ddot{y}\sin\omega t = -\omega^2 a$$

and $$-\ddot{x}\sin\omega t + \ddot{y}\cos\omega t = 0.$$

Show that these equations prove that the acceleration of the point is toward the centre and that it is equal to $\omega^2 a$.

4. A variable quantity θ is given in terms of x by $\theta = a + x \sin\theta$ where a is a constant. Prove that when $x = 0$,

$$\frac{d\theta}{dx} = \sin a \quad \text{and} \quad \frac{d^2\theta}{dx^2} = \sin 2a.$$

Show that an approximate value for θ when x is small is

$$a + x \sin a + \tfrac{1}{2}x^2 \sin 2a.$$

(Cambridge University.)

5. In a triangle ABC the side AB and the distance from C to the mid-point D of AB are accurately measured and found to be 60″ and 30·01″ respectively. The sides AC and BC are more roughly measured and found to be nearly equal. If CD makes an angle θ with CA, prove that if $\delta\theta$ is the amount θ differs from 45° then $\delta(\cot\theta) = \cdot 000333$ and hence C is about 69″ less than a right angle.

CHAPTER XIII

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this chapter we are dealing with an entirely new kind of function. We are going to consider how to differentiate such a function as 10^x: the difference between this and the ordinary algebraic functions so far considered is that here the *variable occurs in the index.*

The formula $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ does not therefore apply to a function of this kind.

To find the value of $\frac{d}{dx}(10^x)$

We shall first of all prove that $\frac{d}{dx}(10^x)$ is equal to $10^x \times$ a numerical factor.

By definition, $$\frac{d}{dx}(10^x) = \operatorname*{Lt}_{h \to 0} \frac{10^{x+h} - 10^x}{h}$$

$$= \operatorname*{Lt}_{h \to 0} 10^x \cdot \left(\frac{10^h - 1}{h}\right)$$

$$= 10^x \cdot \operatorname*{Lt}_{h \to 0} \left(\frac{10^h - 1}{h}\right).$$

Now if $\frac{10^h - 1}{h}$ tends to a limit when $h \to 0$, this limit obviously does not involve x and must be some constant number, call it c.

If the reader wishes to obtain an idea of the numerical value of c, he can do so by evaluating, using logarithm tables, the fraction $\frac{10^h - 1}{h}$ when h is small, say $h = 0{\cdot}01$, and he will then find that c is approximately $2{\cdot}3$. We shall however retain c for the present and employ a different method later for evaluating it.

Thus $\frac{d}{dx}(10^x) = c \cdot 10^x$, where c is a constant.

To find the value of $\frac{d}{dy}(\log_{10} y)$

Let $\log_{10} y = x$, $\therefore\ y = 10^x$.

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(10^x) = c \,.\, 10^x = c \,.\, y,$$

$$\therefore\ \frac{dx}{dy} = \frac{1}{c} \cdot \frac{1}{y},$$

$$\therefore\ \frac{d}{dy}(\log_{10} y) = \frac{1}{c} \cdot \frac{1}{y},$$

which gives us the required result involving the same constant c.

Reversing this equation we write

$$\frac{1}{c}\int \frac{dy}{y} = \log_{10} y + \text{a constant},$$

or
$$\int \frac{dy}{y} = c \log_{10} y + \text{a constant}.$$

We can now obtain the value of c (approximately) by taking a special case.

For we have $\displaystyle\int_1^2 \frac{dy}{y} = \Big[c \log_{10} y\Big]_1^2 = c \log_{10} 2 \triangleq c \times 0{\cdot}301.$

But we showed in Part I, p. 100, that $\displaystyle\int_1^2 \frac{dy}{y} \triangleq 0{\cdot}693.$

$$\therefore\ c \times 0{\cdot}301 \triangleq 0{\cdot}693,$$

$$\therefore\ c \triangleq \frac{0{\cdot}693}{0{\cdot}301} \triangleq 2{\cdot}30.$$

$\therefore$ we have

$$\frac{d}{dx}(10^x) = 2{\cdot}30 \times 10^x \text{ and } \frac{d}{dx}(\log_{10} x) = \frac{1}{2{\cdot}30x},$$

where the numerical factor is approximate.

The presence of this numerical factor complicates these formulae and we therefore proceed to transform them so as to remove it. This transformation is analogous to that made in changing from degrees to circular measure in order to remove the numerical factor that occurs when we evaluate $\frac{d}{dx}\sin x$, etc., if x is measured in degrees.

From the formula $\frac{d}{dx}(10^x) = c \,.\, 10^x$ we have

$$\frac{d}{dx}\left(10^{\frac{x}{c}}\right) = \frac{d\left(10^{\frac{x}{c}}\right)}{d\left(\frac{x}{c}\right)} \times \frac{d\left(\frac{x}{c}\right)}{dx} = c \,.\, 10^{\frac{x}{c}} \times \frac{1}{c} = 10^{\frac{x}{c}}.$$

Now $$10^{\frac{x}{c}} = \left(10^{\frac{1}{c}}\right)^x.$$

If then we put $10^{\frac{1}{c}} = e$, we have $10^{\frac{x}{c}} = e^x$ and our formula becomes

$$\frac{d}{dx}(e^x) = e^x,$$

where $e = 10^{\frac{1}{c}} \doteq 10^{\frac{1}{2\cdot 30}} \doteq 10^{\cdot 435} \doteq 2\cdot 72$.

And further if we now use this number e as the base for our logarithms instead of 10, we remove the awkward numerical factor from the derivative of $\log y$.

Let $$\log_e y = x, \quad \therefore\ y = e^x.$$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x = y,$$

$$\therefore\ \frac{dx}{dy} = \frac{1}{y},$$

$$\therefore\ \frac{d}{dy}(\log_e y) = \frac{1}{y}.$$

The fact that the introduction of this number we have called e simplifies these two formulae, and the reasons why it does so, make the number e play a very important part in all higher mathematics: its value can be computed to as many places of decimals as is wished: to 9 places $e = 2\cdot 718281828$: but just as in the case of π, this decimal neither terminates nor recurs.

In future we shall use the symbol $\log x$ *to mean* $\log_e x$.

It should be noticed that the formula $\int \frac{dx}{x} = \log x$ supplies the

missing result in the general relation $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ which gives the integral of all powers of x except x^{-1}, but fails if $n=-1$.

Example 1.

Find (i) $\frac{d}{dx}(e^{ax})$; (ii) $\frac{d}{dx}\log(ax)$; (iii) $\frac{d}{dx}\log(\sin x)$.

(i) $$\frac{d}{dx}(e^{ax}) = \frac{de^{ax}}{d(ax)} \times \frac{d(ax)}{dx} = e^{ax} \times a = ae^{ax}.$$

(ii) $$\frac{d}{dx}\log(ax) = \frac{d\log(ax)}{d(ax)} \times \frac{d(ax)}{dx} = \frac{1}{ax} \times a = \frac{1}{x},$$

or better $$\frac{d}{dx}\log(ax) = \frac{d}{dx}(\log a + \log x) = \frac{1}{x}.$$

(iii) $$\frac{d}{dx}\log(\sin x) = \frac{d\log(\sin x)}{d(\sin x)} \times \frac{d(\sin x)}{dx} = \frac{1}{\sin x} \times \cos x = \cot x.$$

Example 2.

Find (i) $\int e^{ax}\,.\,dx$; (ii) $\int \frac{dx}{ax+b}$.

(i) Since $\frac{d}{dx}e^{ax} = a\,.\,e^{ax}$, we have $\int e^{ax}\,.\,dx = \frac{1}{a}\,.\,e^{ax} + c$.

(ii) Since $\frac{d}{dx}[\log(ax+b)] = \frac{1}{ax+b} \times a = \frac{a}{ax+b}$,

we have $$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b) + c,$$

or let $\int \frac{dx}{ax+b} = y$ so that $\frac{dy}{dx} = \frac{1}{ax+b}$; put $ax+b=z$;

$$\therefore\ \frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz} = \frac{1}{ax+b} \times \frac{d}{dz}\left(\frac{z-b}{a}\right) = \frac{1}{z} \times \frac{1}{a};$$

$$\therefore\ y = \frac{1}{a}\int \frac{dz}{z} = \frac{1}{a}\log z + c = \frac{1}{a}\log(ax+b) + c.$$

Note in particular $$\int \frac{dx}{b-x} = -\log(b-x) + c.$$

EXAMPLES XIII a

Differentiate with respect to x the expressions in Ex. 1—20:

1. e^{2x}. **2.** $e^x + e^{-x}$. **3.** $e^x - e^{-x}$. **4.** e^{x^2}.

5. e^{ax+b}. **6.** $x e^x$. **7.** $e^{3x}\sin 2x$. **8.** $e^{-2x}\cos 3x$.

9. $\dfrac{e^{ax}}{\sin bx}$. **10.** $\log(3x)$. **11.** $\log(1-x)$. **12.** $\log(x^3)$.

13. $\log\cos x$. **14.** $\log\tan x$. **15.** $\log(x+\sin x)$.

16. $x^2\log x$. **17.** $\log\dfrac{a-x}{a+x}$. **18.** $\log\tan\dfrac{x}{2}$.

19. $\log(\tan x+\sec x)$. **20.** $\log\dfrac{1+\sin x}{1-\sin x}$.

Integrate with respect to x the expressions in Ex. 21—28:

21. e^{2x}. **22.** e^{-3x}. **23.** e^{ax+b}. **24.** $\dfrac{5}{x}$.

25. $\dfrac{3}{4-5x}$. **26.** $\tan x$. **27.** $\cot 3x$. **28.** $\dfrac{\cos x}{1+\sin x}$.

29. Show that $\dfrac{1}{x^2+5x+6}\equiv\dfrac{1}{x+2}-\dfrac{1}{x+3}$ and hence find $\displaystyle\int\dfrac{dx}{x^2+5x+6}$.

30. Find $\dfrac{d}{dx}(x\log x)$ and hence find $\displaystyle\int\log x\,.\,dx$.

31. Use anti-logarithm tables to evaluate $\dfrac{10^h-1}{h}$ when

$$h=0{\cdot}2,\ 0{\cdot}15,\ 0{\cdot}1,\ 0{\cdot}05,\ 0{\cdot}01,\ 0{\cdot}006.$$

32. If $y=\log_{10}x$ and if $\log_{10}e=0{\cdot}4343$, show that $\delta y \doteq \dfrac{0{\cdot}4343\,\delta x}{x}$; and given $\log_{10}5=0{\cdot}6990$, calculate $\log_{10}5{\cdot}1$.

33. If δy is the difference for $1'$ in a table of log-sines where $y=\log_{10}\sin x$, prove that $\delta y\doteq 0{\cdot}0001263\cot x$: [$\log_{10}e=0{\cdot}4343$].

34. Prove that in a table of log-tangents to base 10 the difference of $1'$ in the neighbourhood of $60°$ is approximately $0{\cdot}00029$.

35. If $y=a\,.\,e^{2x}+b\,.\,e^{3x}$, prove that $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=0$.

36. Find the values of a if $y=e^{ax}$ satisfies the equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=0.$$

The Exponential Function

The relation $\dfrac{dy}{dx}=y$ which is satisfied by the function $y=e^x$, called the *exponential function*, is so important that it is necessary to discuss its meaning in detail. The relation states that y or e^x

is a function whose rate of increase per unit increase of x is equal to the function. This functional law has various physical applications:

(i) Consider for example the growth of bacilli in a culture: the rate of increase in the number of bacilli per unit of time is proportional to the number (N) present in the culture at that time (t), in symbols $\frac{dN}{dt} = k \, . \, N$, where k is a constant.

(ii) Newton's law of cooling is of the same type: if the temperature of a body at any time exceeds that of the surrounding air (supposed constant) by θ°, the rate of *decrease* of the temperature per unit increase of time (t) is proportional to θ, in symbols $\frac{d\theta}{dt} = -k\theta$, where k is a positive constant.

But the most familiar example is the increase of a sum of money at Compound Interest.

Suppose £1 invested at 100% per annum comp. int. for x years, the interest being paid n times a year.

At the end of the first period,

$$£\,\frac{1}{n} \text{ interest is paid and the amount is } £\left(1+\frac{1}{n}\right).$$

The second period begins with $£\left(1+\frac{1}{n}\right)$,

$$\text{the interest for this period is } £\,\frac{1}{n}\left(1+\frac{1}{n}\right).$$

$\therefore$ the amount at the end of the second period

$$= \left(1+\frac{1}{n}\right) + \frac{1}{n}\left(1+\frac{1}{n}\right) = £\left(1+\frac{1}{n}\right)^2.$$

Similarly the amount at the end of the third period

$$= \left(1+\frac{1}{n}\right)^2 + \frac{1}{n}\left(1+\frac{1}{n}\right)^2 = £\left(1+\frac{1}{n}\right)^3.$$

And at the end of nx periods or x years, the amount

$$= £\left(1+\frac{1}{n}\right)^{nx}.$$

It should be noticed that the interest added on at the end of each period is a constant fraction $\left(\frac{1}{n}\right)$ of the sum accumulated at the beginning of the period.

The *Amount* therefore advances by a series of jumps which are not equal but increase successively, the rate of increase of the amount per period being $\frac{1}{n} \times$ the sum at the beginning of that period. Now keep the rate of interest fixed at 100 % per annum but suppose the interest paid at more frequent intervals, i.e. make n increase: the jumps become more numerous but each is smaller, and by taking n sufficiently large we can make each jump as small as we please. And in the limit ($n \to \infty$) we can regard the compound interest as being added on all the time (the money increasing like a snowball rolled in the snow) and we obtain a function whose rate of increase per unit time is proportional to its value at that time.

The Amount of £1 in x years at 100 % per annum paid continuously is therefore

$$\underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^{nx}.$$

But the law of growth of this function is precisely the law expressed by the relation $\frac{dy}{dx} = y$ which is satisfied by $y = e^x$, because the increase of the amount in $\frac{1}{n}$th of a year is $\frac{1}{n}$th of the amount at the beginning of that period and therefore the rate of increase *per unit time* (viz. one year) equals the amount at that moment.

Further, when $x=0$, $e^x = e^0 = 1$ and the amount was originally £1.

$\therefore$ initially these two functions are equal.

But if two functions start by being equal and increase at the same rate as each other, they must always remain equal.

$\therefore$ £(e^x) is the amount to which £1 accumulates in x years at 100 % per annum continuous compound interest.

$$\therefore \underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^{nx} = e^x.$$

In particular, if $x=1$ we have $\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^n=e.$

Further it is clear that if the rate of interest is k hundred %, per annum continuous compound interest, the amount of £1 in x years would be $\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{k}{n}\right)^{nx}$.

But since $\frac{d}{dx}(e^{kx})=k\,.\,e^{kx}$, this is the rate of increase which e^{kx} follows: also when $x=0$, $e^{kx}=e^0=1$.

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{k}{n}\right)^{nx}=e^{kx},$$

and in particular putting $x=1$ we have $\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{k}{n}\right)^n=e^k.$

The law embodied in the equation

$$\frac{dy}{dx}=ky,$$

which is satisfied by $y=e^{kx}$, where k is a constant, is therefore called the *Compound Interest Law*: and any function which obeys it is called a Growth Function (e.g. the growth of the bacilli, p. 190).

If we start with £P, the amount at the end of x years at k hundred % per annum continuous compound interest will be £$P\,.\,e^{kx}$: this obeys the same law.

$\therefore$ *$y=P\,.\,e^{kx}$, where P is an arbitrary constant, is the general solution of the relation $\frac{dy}{dx}=ky$.*

This result could be obtained as follows:

$$\frac{dy}{dx}=ky,\ \therefore\ \frac{dx}{dy}=\frac{1}{ky}\ \text{ or }\ x=\frac{1}{k}\int\frac{dy}{y}=\frac{1}{k}\log y+c.$$

$\therefore \log y=k(x-c)$ or $y=e^{kx-kc}=b\,.\,e^{kx}$, where b is a constant.

But when $x=0$, $y=P$; $\therefore P=b$.

$$\therefore y=P\,.\,e^{kx}.$$

The above discussion shows us the rough shape we may expect the graph of e^x to take.

At $x=0$, $e^x=e^0=1$ and $\frac{d}{dx}(e^x)=1$.

$\therefore$ the slope of the curve at this point is 45° upwards.

Also the greater e^x becomes, the more rapidly it increases.

When x is negative and equal to $-z$ say, $e^{-z}=\frac{1}{e^z}$ and is $\therefore$ positive and less than 1.

Its shape is therefore as shown in Fig. 147.

A table of logarithms to base e and a table giving the values of powers of e will be found at the end of this book. By making use of the latter, the graph of e^x can be quickly drawn.

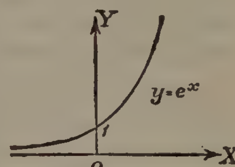

Fig. 147.

EXAMPLES XIII b

1. Plot the graph of $y=e^x$ from $x=-1$ to $x=2$ and then draw the reflection of this curve in the line $y=x$ If the equation of the new curve is $y=f(x)$, what is $f(x)$?

2. Find y if $\frac{dy}{dx}=2y$ and $y=3$ when $x=0$.

3. Find y if $\frac{dy}{dx}+3y=0$ and $y=2$ when $x=0$.

4. Sketch the graph of $y=e^x$, mark the points $P\,(x, y)$, $Q\,(x+h, y+k)$ on it, also the points $P'\,(x', y')$, $Q'\,(x'+h, y'+k')$; prove that

$$\frac{\text{gradient of chord } PQ}{y}=\frac{\text{gradient of chord } P'Q'}{y'}.$$

What does this become when $h \to 0$?

5. What is the area of the figure bounded by the x-axis, the ordinates $x=0$, $x=1$ and the curve $y=e^x$?

6. What is the area of the figure bounded by the x-axis, the y-axis and the curve $y=e^{-x}$?

7. Draw the graph of $y=e^{-x^2}$. [The curve of normal error.]

8. An electric current C is decreasing according to the law $C=ke^{-\frac{t}{\lambda}}$ where k, λ are constants; compare the amount of electricity $\int C\,dt$ passing in time T with the amount passing altogether.

9. The speed of signalling in a submarine telegraph cable is proportional to $x^2 \log\left(\frac{1}{x}\right)$, where x is the ratio of the radius of the core of copper wire to the thickness of the covering. Show that for a maximum speed $x=\frac{1}{\sqrt{e}}$.

10. The amount x of a substance undergoing transformation in a chemical reaction at time t is given by $x=ae^{-kt}$, where a is the amount present at the beginning: show that the velocity of the reaction is proportional to the amount of substance undergoing transformation.

11. *Sketch* the graph of $y=e^{-\frac{x}{2}}\sin x$.
Calculate its gradient when $x=0$ and when $x=\pi$.

12. Find the length of the subtangent at any point of the curve (i) $y=e^x$, (ii) $y=a\,.\,e^{\frac{x}{b}}$.

13. A pane of glass absorbs 2 per cent. of the light incident upon it. How much light will pass through 12 panes of this glass?

14. Does $\log x$ increase more or less rapidly than x as x increases from 0·1 to 10?

15. A particle moves so that when its velocity is v ft./sec. its acceleration is $\frac{1}{5}v$ ft./sec.^{2} Its initial velocity is 5 ft./sec., find its velocity after 10 secs.

16. If the tangent at $P(x, y)$ on the curve $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ makes an angle ψ with the x-axis, prove that $y\cos\psi=a$: and if the tangent cuts the y-axis at T, prove that $PT=\frac{xy}{a}$.

To find $\frac{d}{dx}(a^x)$

Let $a^x=y$, $\therefore\ \log y=\log(a^x)=x\log a$.

$$\therefore \frac{d}{dx}(\log y)=\log a\,\frac{d}{dx}(x)=\log a,$$

but
$$\frac{d}{dx}(\log y)=\frac{d(\log y)}{dy}\times\frac{dy}{dx}=\frac{1}{y}\frac{dy}{dx},$$

$$\therefore \frac{dy}{dx}=y\log a.$$

$$\therefore \frac{d}{dx}(a^x) = a^x \log a,$$

where $\log a \equiv \log_e a$.

Note. When the index is variable, it is often advisable as here to take logarithms before differentiation.

To find $\frac{d}{dx}(\log_a x)$

Let $\log_a x = y$, $\therefore x = a^y$.

$$\therefore \frac{d}{dx}(x) = \frac{d}{dx}(a^y) = \frac{d(a^y)}{dy} \times \frac{dy}{dx} = a^y \log a \frac{dy}{dx},$$

$$\therefore a^y \log a \frac{dy}{dx} = 1.$$

$$\therefore \frac{d}{dx}(\log_a x) = \frac{dy}{dx} = \frac{1}{a^y \log a} = \frac{1}{x \log a}.$$

This result can be obtained more quickly by using the identity

$$\log_a x \equiv \frac{\log_e x}{\log_e a}.$$

When differentiating expressions consisting of two or more factors, it is sometimes helpful to start by taking logarithms: this is illustrated in the next example.

Example 3.

If the pressure p and the volume v of a gas obey the adiabatic law $pv^\gamma = \text{const.}$ where $\gamma \doteq 1{\cdot}414$, find the percentage decrease in p corresponding to a small increase x per cent. in v.

$$pv^\gamma = c, \quad \therefore \log p + \log(v^\gamma) = \log c,$$

or

$$\log p + \gamma \log v = \log c.$$

$$\therefore \frac{1}{p}\frac{dp}{dv} + \frac{\gamma}{v} = 0 \text{ or } \frac{\delta p}{p} \doteq -\gamma \cdot \frac{\delta v}{v}.$$

But

$$\delta v = \frac{vx}{100}, \quad \therefore \frac{\delta p}{p} \doteq -\frac{\gamma x}{100}.$$

$\therefore$ the decrease in p is approximately γx per cent.

Example 4.

The differential equation for damped vibrations of a pendulum is

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + (k^2 + \omega^2)\theta = 0.$$

Show that this is satisfied by $\theta = Ae^{-kt}\sin(\omega t + a)$, where A, a, k, ω are constants.

We have $$\theta \, . \, e^{kt} = A\sin(\omega t + a).$$

Differentiate with respect to t.

$$\therefore e^{kt}\frac{d\theta}{dt} + k\theta \, . \, e^{kt} = A\omega\cos(\omega t + a).$$

Differentiate again.

$$\therefore e^{kt}\frac{d^2\theta}{dt^2} + k \, . \, e^{kt}\frac{d\theta}{dt} + k\frac{d\theta}{dt} \, . \, e^{kt} + k^2\theta \, . \, e^{kt} = -A\omega^2\sin(\omega t + a),$$

$$\therefore e^{kt}\left(\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + k^2\theta\right) = -\omega^2\theta \, . \, e^{kt},$$

$$\therefore \frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + (k^2 + \omega^2)\theta = 0.$$

EXAMPLES XIII c

1. Differentiate with respect to x:

(i) 10^x, (ii) $\log_{10}(2x)$, (iii) $\frac{1}{3^x}$,

(iv) $\log_{10}(1 + x^2)$, (v) $\log_{10}(\cos x)$, (vi) $2^x\log_{10}x$.

2. Integrate with respect to x:

(i) 10^x, (ii) 3^{2x}, (iii) a^{cx}.

3. Evaluate to two significant figures:

(i) $\int_2^3 \frac{dx}{x}$, (ii) $\int_1^2 10^x\,dx$, (iii) $\int_0^1 e^{2x}\,dx$.

4. If $2u = e^x - e^{-x}$ and $2v = e^x + e^{-x}$, prove that

(i) $\frac{du}{dx} = v$, (ii) $\frac{dv}{dx} = u$, (iii) $\frac{d}{dx}(uv) = u^2 + v^2$.

5. If $(1 + x)y = e^x$, prove that $(1 + x)\frac{dy}{dx} = xy$.

6. Find $\frac{d^3y}{dx^3}$ if (i) $y = a \, . \, e^{cx}$, (ii) $y = \log(bx)$.

7. (i) If $pv^\gamma = c$, prove that $v\frac{dp}{dv} + \gamma p = 0$.

(ii) If $y = (x - a)^p(x - b)^q(x - c)^r$, prove that

$$\frac{1}{y}\frac{dy}{dx} = \frac{p}{x - a} + \frac{q}{x - b} + \frac{r}{x - c}$$

8. If $pv=c$, prove that $\int_{v_1}^{kv_1} p\,dv = c\log k$.

9. Prove that $y=e^{2x}(ax+b)$ satisfies the equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0.$$

10. Prove that $t=\frac{1}{2ku}\log\left(\frac{u+v}{u-v}\right)$ satisfies the equation

$$\frac{dv}{dt}=k(u^2-v^2),$$

where k, u are constants.

11. Assuming that $\int e^{ax}\sin bx\,dx=e^{ax}(p\sin bx+q\cos bx)$ where p, q are constants, express p and q in terms of a, b by differentiating and equating coefficients.

12. The equation of the catenary is $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$; if

$$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},$$

and if $s=0$ when $x=0$, express s in terms of x and prove that $y^2=c^2+s^2$.

13. Pressure is brought to bear on a circular plate of radius 1 in.; the pressure at any point distant x ins. from the centre is κe^{-x} tons per sq. in. and the maximum pressure is 40 tons per sq. in. Find the average pressure on the plate in tons per sq. in. $\left(\text{Hint. Find } \frac{d\{(x+1)e^{-x}\}}{dx}\right)$.

14. For a certain curve the ordinates corresponding to $x=0$, 1, 2, 3, 4, 5, 6 are in geometrical progression of ratio 1·2. If $y=1$ when $x=0$, plot the curve. Show that the curve is $y=1{\cdot}2^x$ and express this in the form $y=e^{bx}$.

Find $\frac{dy}{dx}$ when $x=4$ and evaluate $\int_0^4 y\,dx$. (Army.)

15. Prove that $\log x+\frac{1}{x}$ is not less than 1 for any positive value of x.

16. If the deflection of the needle of a damped galvanometer is given by $\theta=Ce^{-\kappa t}\sin\left(\frac{\kappa\pi}{\lambda}t\right)$, prove that

(i) The maximum deflection is when $t=\frac{\lambda}{\kappa\pi}\tan^{-1}\left(\frac{\pi}{\lambda}\right)$

(ii) If Ω is the initial angular velocity, $C=\frac{\Omega\lambda}{\kappa\pi}$.

We shall now indicate some of the numerous physical applications of the Compound Interest Law.

(i) *Newton's Law of Cooling.*

If the temperature of a body is $\theta°$ (Centigrade) above the temperature of the surrounding air which remains constant, the rate of decrease of θ is proportional to θ.

Example 5.

A body in a room of temperature 15° C. starts at a temperature of 75° C. and ten minutes later its temperature is 55° C. What is its temperature after a further five minutes?

Let the temperature be $\theta°$ C. above that of the room after t minutes.

Then we have $\dfrac{d\theta}{dt} = -k\theta$, where k is a constant.

$$\therefore \theta = \theta_0 e^{-kt}.$$

But when $t=0$, $\theta = 75 - 15 = 60$, $\therefore \theta_0 = 60$.

$$\therefore \theta = 60e^{-kt}.$$

But when $t = 10$, $\theta = 55 - 15 = 40$, $\therefore 40 = 60e^{-10k}$.

$$\therefore e^{-10k} = \frac{40}{60} = \frac{2}{3}, \quad \therefore e^{-k} = \left(\frac{2}{3}\right)^{\frac{1}{10}}.$$

[We could of course find k from this equation, but it is unnecessary.]

$\therefore$ when $t = 15$, $\theta = 60e^{-15k} = 60\left[\left(\dfrac{2}{3}\right)^{\frac{1}{10}}\right]^{15} = 60\left(\dfrac{2}{3}\right)^{\frac{15}{10}}$.

$$\therefore \theta = 60\left(\frac{2}{3}\right)^{1\cdot5} = 32\cdot7.$$

$\therefore$ the temperature $= 15 + 32\cdot7 = 47\cdot7°$ C.

(ii) *Coefficient of Expansion.*

If a body is heated, it expands so that the fractional rate of increase of volume per unit increase in temperature is approximately constant; the constant is called the coefficient of cubical expansion.

If the volume is V cu. cms. when the temperature is $\theta°$ C. and the coefficient of expansion is k, we have

$$\frac{\;\frac{dV}{d\theta}\;}{V} = k \quad \text{or} \quad \frac{dV}{d\theta} = kV.$$

$$\therefore V = V_0 . e^{k\theta},$$

where V_0 is the volume at 0° C.

(iii) *Atmospheric Pressure* in still air at constant temperature.

Suppose the pressure at a height of x feet above the ground is p lbs. per sq. in. and that 1 cu. ft. of air at pressure p weighs w lbs.

Consider a column of air on a base of area a sq. ins. between two horizontal planes at heights x and $x + \delta x$ feet.

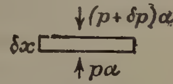

Fig. 148.

The weight of this column is $w \dfrac{a}{144} \delta x$ lbs. But the net supporting force is

$$pa - (p + \delta p)\, a = -\, a\, \delta p \text{ lbs.}$$

$$\therefore\ -a\, \delta p = w \frac{a}{144}\, \delta x,$$

or
$$\frac{dp}{dx} = -\frac{w}{144}.$$

Now the volume v of a given mass of gas $\propto \dfrac{1}{w}$, where w is the density. But by Boyle's law

$$v \propto \frac{1}{p}.$$

$$\therefore\ w \propto p,$$

$$\therefore \frac{w}{p} = \text{const.} = \lambda \text{ say},$$

$$\therefore \frac{dp}{dx} = -\frac{\lambda}{144}\, p.$$

$$\therefore\ p = p_0 e^{-\frac{\lambda x}{144}},$$

where $p = p_0$ when $x = 0$, i.e. on the ground.

If 1 cu. ft. of air at the ground weighs w_0 lbs., we have $\lambda = \dfrac{w_0}{p_0}$ and our equation becomes

$$\frac{p}{p_0} = e^{-\frac{w_0 x}{144 p_0}}.$$

(iv) *Leakage in an electric condenser.*

If a condenser contains a charge, the leakage at any time is proportional to the charge.

If the initial charge is Q_0 and if after t secs. the charge falls to Q, we have

$$\frac{dQ}{dt} = -\lambda Q, \text{ where } \lambda \text{ is a constant.}$$

$$\therefore\ Q = Q_0 \cdot e^{-\lambda t}$$

(v) *Tension of a belt passing over a rough circular pulley.*

Consider a small portion PQ of the belt which subtends an angle $\delta\psi$ radians at the centre O of the pulley. The belt is on the point of sliding in the direction $P \rightarrow Q$. The angle between the tangents PKH and KQ is $\delta\psi$. We can replace $T + \delta T$ along KQ by $(T + \delta T) \sin \delta\psi$ perpendicular to KH and $(T + \delta T) \cos \delta\psi$ along KH.

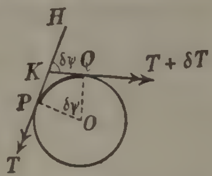

Fig. 149.

$\therefore$ for small values of $\delta\psi$ the effect of the tensions on PQ is represented by

(i) along PH, $(T + \delta T) \cos \delta\psi - T \doteq \delta T$ since $\cos \delta\psi \doteq 1$,

(ii) perpendicular to PH, $(T + \delta T) \sin \delta\psi \doteq T\delta\psi + \delta T \delta\psi \doteq T\delta\psi$.

But this is balanced by the normal reaction R and the friction μR, if μ is the coefficient of friction.

$$\therefore\ T\delta\psi \doteq R \text{ and } \delta T \doteq \mu R,$$

$$\therefore\ \delta T \doteq \mu T \delta\psi,$$

$$\therefore\ \frac{dT}{d\psi} = \mu T.$$

$$\therefore\ T = T_0 \cdot e^{\mu\psi},$$

where T_0 is the tension when $\psi = 0$.

Example 6.

A cable makes one complete turn round a circular post, the coefficient of friction being $\frac{3}{4}$. One end is held by a man who can exert a force of 40 lbs.; what strain can the other end support?

Here $\mu = \frac{3}{4}$, $T_0 = 40$, $\psi = 2\pi$ (one complete turn).

$$\therefore\ T = 40e^{\frac{3\pi}{2}}$$

$$= 4440 \text{ lbs. or nearly 2 tons.}$$

Example 7.

A vertical column 15 feet high has a plane horizontal top of area 1·5 sq. ins. which carries a weight of 30 lbs. The column is so shaped that the pressure per sq. in. across every horizontal section is constant for all sections. If the material weighs $\frac{1}{4}$ lb. per cu. in., find the area of the base. If the column is a solid of revolution, what curve must be used to generate it?

The pressure on the top is 30 lbs. per 1·5 sq. ins. which equals 20 lbs. per sq. in.

$\therefore$ the pressure across every horizontal section must be 20 lbs. per sq. in.

Let the area of a section x ins. from the top be A sq. ins. Then the forces on the portion between two sections at distances x ins. and $x+\delta x$ ins. from the top are shown in Fig. 150. For the support from below is $20(A+\delta A)$ lbs. and the total pressure from above is $20A$ lbs. and the weight of the section (volume $A\,\delta x$ cu. ins.) is $\frac{1}{4}A\,\delta x$ lbs.

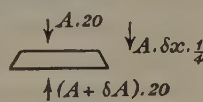

Fig. 150.

[Note that the weight really lies between $\frac{1}{4}A\,\delta x$ and $\frac{1}{4}(A+\delta A)\,\delta x$, but the product $\delta A\,\delta x$ disappears in the limit.]

$\therefore$ the net support $=20(A+\delta A)-20A=20\delta A$ lbs.

$$\therefore\ 20\delta A \triangleq \tfrac{1}{4}A\,\delta x \text{ or } \frac{\delta A}{\delta x}\triangleq\frac{A}{80}.$$

$\therefore$ in the limit $\dfrac{dA}{dx}=\dfrac{A}{80}$, $\therefore\ A=A_0 e^{\frac{x}{80}}$.

But when $x=0$, $A=1·5$. $\therefore\ A_0=1·5$.

$$\therefore\ A=1·5e^{\frac{x}{80}}.$$

Now the height $=15$ feet $=180$ ins.

$$\therefore \text{ area of base}=1·5e^{\frac{180}{80}}=1·5e^{2·25} \text{ sq. ins.}$$
$$=1·5\times 9·49=14·2 \text{ sq. ins.}$$

Suppose the radius of the circular section at depth x ins. is y ins., then

$$\pi y^2=1·5\,e^{\frac{x}{80}}.$$

$$\therefore\ y=\sqrt{\frac{1·5}{\pi}}\,e^{\frac{x}{160}}.$$

$$\therefore\ y=0·69\,e^{\frac{x}{160}}.$$

If this curve is revolved about the x-axis, the axis of the column, the required solid will be generated.

EXAMPLES XIII d

1. A body starts with velocity u ft./sec. and moves with an acceleration kv ft./sec.^{2} where v is its speed: find the distance it travels in t secs.

2. The temperature of a liquid in a room of constant temperature 20° C. is observed to be 70° C. After 5 minutes it is 60°. What will it be after another 30 minutes?

3. The current C at time t in a conductor is falling off according to the law $\frac{dC}{dt}+kC=0$, where k is a constant: initially $C=2$ and after $\frac{1}{10}$ sec. $C=1$; find k and the time taken for the current to fall to 0·01.

4. When a shell is moving with velocity v ft./sec., the air resistance causes a retardation $\frac{1}{10}(v-500)$ ft./sec.^{2} as long as $v>1000$. Its muzzle velocity is 2500 ft./sec.: what is its velocity after 5 seconds?

5. A tank which starts full is being emptied so that the rate at which the water runs out is proportional to the amount left in. If half the water runs out in 10 minutes, what fraction will remain after a quarter of an hour from the start?

6. A body of volume 1000 cu. cms. is at temperature 20° C.; when the temperature rises 10° C. the volume increases by 1 cu. cm. What is its volume at 100° C.?

7. For a condenser of capacity K charged with a quantity Q of electricity at potential V, if R is the resistance in the circuit, we have

$$Q=KV,\ \ V=RC,\ \ C=-\frac{dQ}{dt}.$$

If $V=V_0$ when $t=0$, prove that $R=t\div\left\{K\log\frac{V_0}{V}\right\}$.

8. A bullet is shot vertically upwards with velocity u ft./sec.; owing to air resistance its retardation is $g(1+kv^2)$ ft./sec.^{2} when its velocity is v ft./sec. Prove that the greatest height it reaches is $\frac{1}{2gk}\log(1+ku^2)$ feet.

9. A long metal tube, internal radius 1 cm. and external radius 2 cms., is filled with steam kept at temperature 125° C., the temperature in the metal at r cms. from the axis of the tube is T° C. where $\frac{dT}{dr}=-\frac{100}{r}$; find the temperature at the outer surface.

10. How many turns must be taken round a rough circular post, coefficient of friction $\frac{1}{4}$, so that a pull of 2 tons can be resisted by a force of 60 lbs.?

11. A rotating flywheel is subject to a frictional couple proportional to its angular velocity: its initial velocity is ω_1, and after 1 second it is ω_2; find its angular velocity after t seconds.

12. A jar of water at 15° C. is placed in a temperature of $-12°$ C. and its temperature falls 5° in 8 minutes. How long will it be before ice begins to form?

13. A raindrop falls with acceleration $g - \frac{v}{2}$ ft./sec.², where v ft./sec. is its velocity. What is its limiting velocity and how far does it fall in 10 secs. from rest?

14. If the vapour pressure P and the absolute temperature T are connected by the equation $P = aT^n e^{\frac{b}{T}}$, where a, b, n are constants, find the value of T for which P is stationary. What is the condition that this value of P is a maximum?

15. One end of a metal bar in air at temperature 0° C. is kept a constant temperature T° C.; heat flows along the bar so that at a distance x feet from that end the temperature $\theta°$ is given by $\frac{d^2\theta}{dx^2} = n^2\theta$, where n is a constant: show that $\theta = Ae^{nx} + Be^{-nx}$ and use the fact that if the bar is very long the temperature at the far end remains at 0° C. to find A, B.

16. In a chemical change the rate of decomposition of a substance is proportional to the amount C that remains at any time t, and the percentage increase of pressure p is proportional to the percentage decrease of the substance: the initial values of C, p are C_0, p_0; express C and p in terms of t and the constants of variation, and find a relation between p, C.

17. A long vertical rod carries a weight of 5000 lbs. at its lower end; each horizontal section of the rod is of such a size that the stress across every section is 2000 lbs. per sq. in. and the material weighs $\frac{1}{4}$ lb. per cu. in. What is the area of the cross-section (i) at the lower end, (ii) 50 feet above the lower end, (iii) 200 feet above the lower end?

18. A rocket of weight 10 lbs. is fired vertically upwards starting from rest; the effect of the gradual burning of the charge is to produce an acceleration of $\frac{15g}{10-t}$ ft./sec.^{2} after t secs. The charge is consumed after 5 secs.; what is then the rocket's velocity? Take $g=32$.

19. When the voltage in a circuit of self-induction L suddenly changes from 0 to V, the current C after time t is given by

$$V=RC+L\frac{dC}{dt};$$

prove that

$$C=\frac{V}{R}+Ae^{-\frac{Rt}{L}},$$

and find A if $C=0$ when $t=0$.

20. A body A of mass 40 lbs. lies on a rough horizontal plane AC (coefficient of friction $\frac{1}{4}$) and is pulled slowly along the plane by a rope passing over a smooth pulley B: BC is vertical and equals 5 feet. When $AC=x$ feet, the work δW ft.-lbs. done in a small displacement δx ft. is given by

$$\delta W \doteq -10\delta x \bigg/ \left(1+\frac{5}{4x}\right).$$

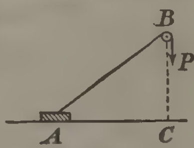

Fig. 151.

Find the work done if the initial and final lengths of AC are 20 feet and 5 feet.

CHAPTER XIV

DIFFERENTIALS AND GENERAL INTEGRATION

Differentials

LET the tangent at any point P of the curve $y=f(x)$ cut at S the ordinate of another point Q on the curve. Then $NM=PR$ being δx, RQ will be δy. The tangent of the angle SPR will be $\frac{dy}{dx}$ or $f'(x)$, hence $\frac{RS}{PR}=f'(x)$, i.e. $RS=f'(x)\,PR$. The increments RS and PR are known as *differentials* and are called dy and dx respectively. Hence $dy=f'(x)\,dx$ or $\left[\frac{d}{dx}f(x)\right].dx$ or $\left[\frac{d}{dx}(y)\right].dx$; dy and dx are therefore two quantities such that the ratio between them is the **differential coefficient** $\frac{dy}{dx}$.

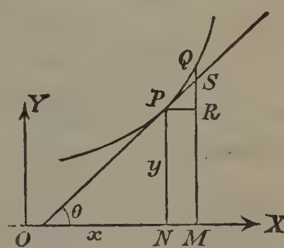

Fig. 152.

Now dx and δx are quite arbitrary but when δx and dx are fixed it is possible to calculate δy and dy.

When $dx=\delta x=NM$, then dy is RS but δy is RQ; $\therefore$ δy is not equal to dy, but the ratio $\frac{dy}{\delta y}$ can be made as nearly equal to 1 as we please by taking δx sufficiently small.

The most useful application of differentials arises in the methods of Integration which will be discussed in the next section.

Product and Quotient Formulae in terms of Differentials

We have shown that $\frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$.

Now $du = \frac{du}{dx}\,dx$, $dv = \frac{dv}{dx}\,dx$, and $d\,(uv) = \frac{d}{dx}(uv)\,dx$;

$$\therefore\ d\,(uv) = u\,dv + v\,du.$$

Similarly $$\frac{d\,(uvw)}{uvw} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w}.$$

This result may be most easily obtained by using logarithms, thus:

$$\log\,(uvw) = \log u + \log v + \log w,$$

$$\therefore\ \frac{d}{dx}\log\,(uvw) = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w\,;$$

i.e. $$\frac{1}{uvw}\,\frac{d\,(uvw)}{dx} = \frac{1}{u}\,\frac{du}{dx} + \frac{1}{v}\,\frac{dv}{dx} + \frac{1}{w}\,\frac{dw}{dx},$$

or $$\frac{d\,(uvw)}{uvw} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w}$$

Similarly $$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}.$$

$$\therefore\ d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

In other words the relation between Differentials is obtained by treating $\frac{dy}{dx}$ as if it were a fraction; thus if $y = x^2$ then $\frac{dy}{dx} = 2x$ and $dy = 2x\,dx$.

Example 1.

If $y = 3\,(2x+1)^2$, find dy.

$$\frac{dy}{dx} = 12\,(2x+1),\quad \therefore\ dy = 12\,(2x+1)\,dx\,;$$

or $$d\,[3\,(2x+1)^2] = 12\,(2x+1)\,dx.$$

Example 2.

Find the differential of $\sin\theta$.

Since $$\frac{d\,(\sin\theta)}{d\theta} = \cos\theta,\quad \therefore\ d\,(\sin\theta) = \cos\theta\,.\,d\theta.$$

Application of Differentials

Errors. Since δy is approximately equal to the differential dy when δx is small, the calculation of the effect of errors may conveniently be carried out by using differentials.

Example 3.

The hypotenuse of a triangle is calculated from $c^2=a^2+b^2$ when $a=8{\cdot}21$ cms., $b=4{\cdot}03$ cms. measured to two places of decimals.

Find the greatest possible error in c.

Since $$c^2=a^2+b^2 \text{ and } d(c^2)=2c\,dc,$$
we have $$2c\,dc=2a\,da+2b\,db,$$
i.e. $$dc=\frac{a}{c}\,da+\frac{b}{c}\,db.$$

By calculation $c=9{\cdot}145$, also da and db may both be $\cdot 005$;

$$\therefore\ dc=\frac{8{\cdot}21\times{\cdot}005+4{\cdot}03\times{\cdot}005}{9{\cdot}145}=\frac{12{\cdot}24\times{\cdot}005}{9{\cdot}145}=0{\cdot}0067.$$

It will be found convenient to use Differentials when dealing with more than two variables.

Example 4.

Find the least area of canvas that can be used to construct a conical tent whose cubical capacity is 800 cu. ft.

We have $V=800=\frac{1}{3}\pi r^3\cot\theta$ ....(i),

and A the area of canvas $=\pi rl$

$=\pi r^2\operatorname{cosec}\theta$ ....(ii).

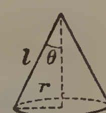

Fig. 153.

Instead of expressing A in terms of r or θ from (i) proceed as follows:

From (i) $r^3\cot\theta=$const., $\therefore\ d(r^3\cot\theta)=0$,

i.e. $$r^3(-\operatorname{cosec}^2\theta\,d\theta)+\cot\theta\,(3r^2dr)=0 \quad ....(iii).$$

From (ii) since A is to be a minimum $dA=0$,

i.e. $$\pi\,[r^2(-\operatorname{cosec}\theta\cot\theta\,d\theta)+\operatorname{cosec}\theta\,2r\,dr]=0 ....(iv).$$

Now eliminate $d\theta$ and dr from (iii) and (iv);

$$\frac{d\theta}{dr}=\frac{3r^2\cot\theta}{r^3\operatorname{cosec}^2\theta}=\frac{2r\operatorname{cosec}\theta}{r^2\operatorname{cosec}\theta\cot\theta},$$

$$\therefore\ 3\cot^2\theta=2\operatorname{cosec}^2\theta=2+2\cot^2\theta,$$

$$\therefore\ \cot^2\theta=2,\quad \theta=35^\circ\,15',\quad r=8{\cdot}14,\quad \underline{A=361 \text{ sq. ft.}}$$

EXAMPLES XIV a

1. Write down the differentials of the following:

(i) ax^2+b; (ii) a^2-2x^2; (iii) $3x^2-5x+2$;

(iv) x; (v) $x^{\frac{1}{2}}$; (vi) $x^{-\frac{1}{2}}$;

(vii) $x+\frac{1}{x}$; (viii) $\cos\theta$; (ix) $\sec\theta$.

(x) $\tan(2x+3)$; (xi) $\tan^{-1}x$; (xii) $\sec^2\theta$;

(xiii) $\sin(3x+5)$; (xiv) $\operatorname{cosec}(2a+x)$; (xv) $\sec(a-x)$;

(xvi) $\log x$; (xvii) $\log(ax+b)$; (xviii) $x^{\frac{1}{2}}+x^{-\frac{3}{2}}$;

(xix) e^{ax}; (xx) e^{-3x}.

2. Write down the functions of which the following are the differentials:

(i) dx; (ii) $(2x+3)\,dx$; (iii) $x^{-\frac{1}{2}}\,dx$;

(iv) $\frac{dv}{v^{1\cdot4}}$; (v) $\frac{du}{u^{\frac{3}{2}}}$; (vi) $\frac{du}{u+1}$;

(vii) $\frac{x^2-3}{x^2}\,dx$; (viii) $\sec\theta\tan\theta\,d\theta$; (ix) $\operatorname{cosec}^2\theta\,d\theta$.

3. Find the error in the area of a triangle calculated from $\frac{1}{2}ab\sin C$ when $a=6{\cdot}4$, $b=5{\cdot}8$ and $C=36°\,24'$ if there is an error of $2'$ in C

4. An error of 2 °/。 is made in measuring an angle θ as 48·4°. Find the percentage error in calculating the value of $\sin\theta+\cos\theta$.

5. A beam supported at its ends, and loaded at the mid-point with a weight W, sags through a distance d given by $d=WK\frac{l^3}{bt^3}$, where K is constant.

If $l=10$ ft., $b=6$ ins., $t=8$ ins. and each measurement may be 1 °/。 in error, find the possible percentage error in the deflection. (Take logs first.)

6. The weight of copper (W) deposited in t secs. by a current C passing through copper sulphate is $W=Cte$, where e is the electro-chemical equivalent of copper. If the current can be read to a °/。, the weight to b °/。, the time to c °/。, find the possible percentage error in calculating e.

7. A long rectangular strip of paper has a width AB of 4 ins. One corner A is folded over so that A comes on the long edge through B at A'. If the crease is CD where C is on AB, prove that $x(\cos 2\theta+1)=4$ where $x=AC$ and $\theta=\angle ADC$. Hence find the smallest area of the ΔACD.

8. Find the shape of the cylinder with the greatest volume for a given surface area A.

9. In using the formula $c^2=a^2+b^2-2ab\cos C$ for finding c, the errors in a and b are both $+2$ °/₀, but C is correct. Find the resultant percentage error in c.

10. Find the possible error in C if the formula for $\cos C$ is used when $a=64$, $b=27$, $c=43$ and there may be an error either way of 1 °/₀ in each measurement.

11. A crank $CP\,(r)$ rotates about C while the connecting rod $PQ\,(l)$ moves so that Q travels along a line QC. If, when CP makes an angle θ with CQ and PQ makes an angle ϕ with QC, the angular velocity of QP is ω' and the angular velocity of CP is ω, prove that the velocity of Q is $-l\omega'\sin\phi-r\omega\sin\theta$.

Also prove $\dfrac{\omega'}{\omega}=\dfrac{r\cos\theta}{l\cos\phi}$ and $\dfrac{\text{vel. of } Q}{\text{vel. of } P}=\dfrac{\sin(\theta+\phi)}{\cos\phi}$.

12. A is a fixed point 1 foot above a fixed horizontal line MN. A rod AB, 6 ins. long, swings below A through $\theta°$ on the left of the vertical and an arm BC 2 feet long moves so that C slides along MN to the right of the vertical. Find the velocity of C when $\theta=15°$, if the velocity of B is then 4 ins. per sec.

13. Given $t=2\pi\sqrt{\dfrac{l}{g}}$, find the percentage change in t if l is increased by 1 °/₀.

Virtual Work

If a body or system of bodies is in equilibrium the work done by the external forces in any small displacement consistent with the geometrical conditions, i.e. any virtual displacement, is zero.

If the dimension, which is to be varied, is measured from a fixed line, the virtual work is positive when the force and the dimension are both measured in the same direction.

Example 5.

Five light rods are freely jointed together so as to form a square $ABCD$ and one diagonal BD. It is suspended from A and a weight W hangs from C; show that the thrust in $BD=W$.

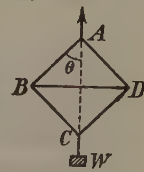

Fig. 154.

Let $\angle BAC=\theta$; then if a is the side of the square, $AC=2a\cos\theta$ (measured from A), $BD=2a\sin\theta$ (measured on both sides from the vertical AC).

By the principle of virtual work if T is the thrust in BD

$$Wd(AC)+Td(BD)=0;$$

i.e. $-Wa\sin\theta+Ta\cos\theta=0$,

$$\therefore\ T=W\tan\theta=W \qquad \text{since } \theta=\frac{\pi}{4}.$$

Example 6.

A string ACB passes over two pulleys A and B in a horizontal line 4 feet apart, it has weights of 5 lbs. at each end and 6 lbs. at C between the pulleys. Find the position of equilibrium.

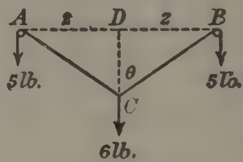

Fig. 155.

If $\angle DCB=\theta$ we have $DC=2\cot\theta$ and $BC=2\operatorname{cosec}\theta$.

If $2l$ is the length of string and z the vertical distance of the 5 lbs. below AB, then $z=l-2\operatorname{cosec}\theta$ and $dz=-2d(\operatorname{cosec}\theta)$.

By virtual work $6d\,(2\cot\theta)+10dz=0$,

i.e. $6d\,(2\cot\theta)-20d\,(\operatorname{cosec}\theta)=0$,

$$\text{i.e. } -12\operatorname{cosec}^2\theta+20\operatorname{cosec}\theta\cot\theta=0,$$

$$\therefore\ \cos\theta=\tfrac{3}{5} \text{ for equilibrium.}$$

EXAMPLES XIV b

1. Four equal uniform rods each of weight W are smoothly hinged to form a rhombus $ABCD$. B and D are kept apart by a rod of weight w when the whole is suspended from A. If the angle DAB is $2a$, find the thrust in BD.

2. A ladder stands on a smooth floor with its legs both inclined at an angle θ to the vertical and their mid-points joined by a horizontal cord. If a weight W is placed on the top, find the additional tension in the cord.

3. A tripod made of three equal uniform rods freely jointed together stands on a smooth floor with the lower ends of the rods connected by strings equal in length to the rods. Find the tension in the strings if each rod weighs 2 lbs. and find also what the tension will be if 4 lbs. is placed on the vertex of the tripod.

4. A regular hexagon $ABCDEF$ of 6 equal rods, each of weight W, is freely jointed together. If it rests in a vertical plane with AB on a horizontal table, find the tension in a light string joining C to F.

Methods of Integration

Integration is largely a tentative process, but there are certain well-defined types of Integrals which admit of simple treatment which we now proceed to illustrate.

I. Integration at Sight

The following gives a list of Integrals which follow at once from the fact that Integrating is the reverse operation of Differentiating.

y	$\dfrac{dy}{dx}$	Integral
x^n	nx^{n-1}	$\int x^n\,dx=\dfrac{x^{n+1}}{n+1}+c \qquad (n \neq -1)$
$(ax+b)^n$	$an\,(ax+b)^{n-1}$	$\int (ax+b)^n\,dx=\dfrac{(ax+b)^{n+1}}{a\,(n+1)}+c$
$\sin(ax+b)$	$a\cos(ax+b)$	$\int \cos(ax+b)\,dx=\dfrac{\sin(ax+b)}{a}+c$
$\cos(ax+b)$	$-a\sin(ax+b)$	$\int \sin(ax+b)\,dx=-\dfrac{\cos(ax+b)}{a}+c$
$\tan(ax+b)$	$a\sec^2(ax+b)$	$\int \sec^2(ax+b)\,dx=\dfrac{\tan(ax+b)}{a}+c$
$\cot(ax+b)$	$-a\,\mathrm{cosec}^2(ax+b)$	$\int \mathrm{cosec}^2(ax+b)\,dx=-\dfrac{\cot(ax+b)}{a}+c$
e^{ax}	ae^{ax}	$\int e^{ax}\,dx=\dfrac{e^{ax}}{a}+c$
$\log_e x$	$\dfrac{1}{x}$	$\int \dfrac{1}{x}\,dx=\log_e x+c$
a^x	$a^x\log_e a$	$\int a^x\,dx=\dfrac{a^x}{\log_e a}+c$

Example 7.

Find $\int(2x+3)^3\,dx$.

As a tentative solution we write $(2x+3)^4$. Mental differentiation of this gives $8\,(2x+3)^3$.

$$\therefore\ \int(2x+3)^3\,dx=\tfrac{1}{8}\,(2x+3)^4+c.$$

Example 8.

Integrate $\sin(3x+2)$.

We try $\cos(3x+2)$; mental differentiation gives $-3\sin(3x+2)$.

$$\therefore \int \sin(3x+2)\,dx = -\tfrac{1}{3}\cos(3x+2)+c.$$

Example 9.

Find $\int e^{-3x}\,dx$.

We try e^{-3x} which when differentiated gives $-3e^{-3x}$.

$$\therefore \int e^{-3x}\,dx = -\tfrac{1}{3}e^{-3x}+c$$

Example 10.

Find $\int \frac{dx}{2-3x}$.

We try $\log_e(2-3x)$ which when differentiated gives $\frac{1}{2-3x}\times(-3)$.

$$\therefore \int \frac{dx}{2-3x} = \frac{\log_e(2-3x)}{-3}+c.$$

Note that the Denominator here must be *linear*, for $\frac{d}{dx}(\log_e x^2)$ is not $\frac{1}{x^2}$.

EXAMPLES XIV c

Find

1. $\int (x+4)^2\,dx$.
2. $\int (2-x)^3\,dx$.
3. $\int (2x+3)^{\frac{1}{2}}\,dx$.
4. $\int \frac{dx}{(2x+1)^3}$.
5. $\int \sqrt{2x+5}\,dx$.
6. $\int \sqrt{ax+b}\,dx$.
7. $\int \cos(2x-5)\,dx$.
8. $\int \sin(4-x)\,dx$.
9. $\int \sec^2(nx+a)\,dx$.
10. $\int \sin^2 x\,d(\sin x)$.
11. $\int \sin^2 x\cos x\,dx$.
12. $\int \tan x\,d(\tan x)$.
13. $\int \tan x\sec^2 x\,dx$.
14. $\int \cot x\,\mathrm{cosec}^2 x\,dx$.
15. $\int e^{-ax}\,dx$.
16. $\int e^{(ax+b)}\,dx$.
17. $\int \frac{dx}{2x-3}$.
18. $\frac{dx}{ax+b}$.
19. $\int \frac{dx}{2-x}$.
20. $\int x^{0.4}\,dx$.
21. $\int_2^1 \frac{dv}{v^{1.4}}$.
22. $\int x^{-0.4}\,dx$.
23. $\int (x+1)\sqrt{(x^2+2x+3)}\,dx$.
24. $\int \frac{\sin x}{(1+2\cos x)^2}\,dx$.
25. $\int xe^{-x^2}\,dx$.

II. **Change of variable by substitution**

If $z=\int f(x)\,dx$, then $\dfrac{dz}{dx}=f(x)$.

Now substitute $x=\phi(u)$, i.e. $\dfrac{dx}{du}=\phi'(u)$, and express the integral in terms of u as the variable

$$\frac{dz}{du}=\frac{dz}{dx}\times\frac{dx}{du}=\frac{dz}{dx}\times\phi'(u)$$

$$=f(x)\times\phi'(u);$$

$$\therefore\ z=\int f(x)\,\phi'(u)\,du.$$

That is to say, we may write for dx **the differential** $\phi'(u)\,du$ **and then integrate with respect to** u**.**

We shall consider two special cases:

(*a*) *Reduction to the form* $\int x^n\,dx$.

Example 11.

Find $\int x\sqrt{x^2+1}\,dx$.

Let $z=\int x\sqrt{x^2+1}\,dx$, then $\dfrac{dz}{dx}=x\sqrt{x^2+1}$. Substitute $u=x^2+1$, then $\dfrac{du}{dx}=2x$.

Changing from $\dfrac{dz}{dx}$ to $\dfrac{dz}{du}$ we have

$$\frac{dz}{du}=\frac{dz}{dx}\times\frac{dx}{du}$$

$$=x\sqrt{x^2+1}\times\frac{1}{2x}=\frac{\sqrt{x^2+1}}{2}=\frac{u^{\frac{1}{2}}}{2};$$

$$\therefore\ z=\tfrac{1}{2}\int u^{\frac{1}{2}}\,du+c=\tfrac{1}{3}u^{\frac{3}{2}}+c$$

$$=\frac{(x^2+1)^{\frac{3}{2}}}{3}+c.$$

The correctness of this result should be tested by finding $\dfrac{dz}{dx}$ from it.

We may however proceed more shortly as follows:

By direct application of the result proved above that

$$\int f(x)\,dx = \int f(x)\,\phi'(u)\,du,$$

we have, by putting $u = x^2+1$,

$$\frac{du}{dx} = 2x \quad \text{or} \quad du = 2x\,dx;$$

$$\therefore \int x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\,(x\,dx)$$

$$= \int u^{\frac{1}{2}}\frac{du}{2} = \tfrac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{(x^2+1)^{\frac{3}{2}}}{3} + c.$$

Note. When we change the variable in a definite integral we must of course also change the limits, and since the result of a definite integral is a function of the limits and not a function of the variable, there is no need to substitute back to the original variable. This will be clear if we integrate the last example from $x=0$ to $x=1$. Then

$$\int_0^1 x\sqrt{x^2+1}\,dx = \tfrac{1}{2}\int_1^2 u^{\frac{1}{2}}\,du.$$

For since $u = x^2+1$, when $x=0$, u will equal 1,

and when $x=1$, u will equal 2.

$\therefore$ the limits for u are from 1 to 2.

$$\therefore \text{integral} = \tfrac{1}{3}\Big[u^{\frac{3}{2}}\Big]_1^2 = \frac{(\sqrt{8}-1)}{3}$$

(*b*) *Reduction to the form* $\int \frac{dx}{x}$

Example 12.

Find $\int \frac{x}{x^2+1}\,dx$.

We notice that the numerator x equals the derivative of the denominator multiplied by a constant; this suggests the form $\int \frac{dx}{x}$.

Put $u = x^2+1$, then $du = 2x\,dx$;

$$\therefore \int \frac{x}{x^2+1}\,dx = \tfrac{1}{2}\int \frac{du}{u} = \tfrac{1}{2}\log_e u + c$$

$$= \tfrac{1}{2}\log_e(x^2+1) + c.$$

Example 13.

Find $\int \tan x\,dx$.

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x} = -\log_e \cos x + c$$
$$= \log_e \sec x + c.$$

EXAMPLES XIV d

Evaluate:

1. $\int x\sqrt{3-x^2}\,dx.$ 2. $\int \sqrt{a^2-x^2}\,x\,dx.$ 3. $\int \frac{dx}{\sqrt{1-x}}.$

4. $\int \frac{x\,dx}{\sqrt{1-x^2}}.$ 5. $\int \frac{x^2\,dx}{\sqrt{a^2+x^3}}.$ 6. $\int 2\pi y\sqrt{y^2+a^2}\,dy.$

7. $\int_0^1 \frac{dx}{\sqrt{2-x}}.$ 8. $\int_2^3 \frac{x\,dx}{x^2-1}.$ 9. $\int \frac{t^2\,dt}{3+t^3}.$

10. $\int \frac{3ax\,dx}{b^2+a^2x^2}.$ 11. $\int \frac{1}{3-x}\,dx.$ 12. $\int \frac{1}{a-bx}\,dx.$

13. $\int \cot x\,dx.$ 14. $\int \frac{(3x^2-2x)\,dx}{x^3-x^2+1}.$ 15. $\int \frac{1}{x}\log x\,dx.$

16. $\int x^3\sqrt{(1+x^4)}\,dx.$ 17. $\int \sin^3 x\cos x\,dx.$ 18. $\int_0^{\frac{\pi}{3}} \frac{\sin\theta\,d\theta}{3+4\cos\theta}.$

19. $\int \tan^5\theta\sec^2\theta\,d\theta.$

III. Powers of $\sin x$ or $\cos x$

In order to integrate powers of $\sin x$ or $\cos x$ or products of $\sin x$ and $\cos x$ it is usually necessary to express them as functions of the first degree, using multiple angles.

Example 14.

Find (i) $\int \sin^2 x\,dx$; (ii) $\int \cos^3 x\,dx$.

(i) We have $\cos 2x = 1 - 2\sin^2 x$ or $\sin^2 x = \frac{1-\cos 2x}{2}$,

$$\therefore \int \sin^2 x\,dx = \tfrac{1}{2}\int (1-\cos 2x)\,dx = \tfrac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c.$$

(ii) We have $\cos 3x = 4\cos^3 x - 3\cos x$ or $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$,

$$\therefore \int \cos^3 x\,dx = \tfrac{1}{4}\int (\cos 3x + 3\cos x)\,dx = \tfrac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c.$$

IV. Trigonometrical substitutions

It is often possible to simplify expressions by using a trigonometrical substitution; e.g. if the expression involves $\sqrt{(a^2-x^2)}$, and if we put $x=a\sin\theta$, we have

$$\sqrt{(a^2-x^2)}=\sqrt{(a^2-a^2\sin^2\theta)}=\sqrt{(a^2\cos^2\theta)}=a\cos\theta;$$

or if the expression involves (a^2+x^2), it may be useful to put $x=a\tan\theta$, then $a^2+x^2=a^2+a^2\tan^2\theta=a^2\sec^2\theta$;

or more generally in $\sqrt{[a^2-(x+b)^2]}$, put $x+b=a\sin\theta$.

The cases $\sqrt{(x^2-a^2)}$ and $\sqrt{x^2+a^2}$ will be considered later (see p. 272).

Example 15.

Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{(9-x^2)}}$.

Put $\qquad x=3\sin\theta, \ \therefore\ dx=3\cos\theta\, d\theta,$

and $\qquad \sqrt{(9-x^2)}=\sqrt{(9-9\sin^2\theta)}=\sqrt{(9\cos^2\theta)}=3\cos\theta.$

Also when $x=0$, $\theta=0$ and when $x=3$, $\sin\theta=1$ and $\theta=\frac{\pi}{2}$.

$$\therefore\ \text{expression}=\int_0^{\frac{\pi}{2}}\frac{3\cos\theta\, d\theta}{3\cos\theta}=\int_0^{\frac{\pi}{2}}d\theta=\Big[\theta\Big]_0^{\frac{\pi}{2}}=\frac{\pi}{2}.$$

Note. This integral could be evaluated more shortly by using the fact

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]=\frac{1}{\sqrt{(a^2-x^2)}}.$$

Example 16.

Evaluate $\qquad \displaystyle\int\sqrt{a^2-x^2}\,dx.$

Here x is less than a, so we try $x=a\sin\theta$ then $dx=a\cos\theta\, d\theta$.

$$\begin{aligned}\therefore\ \int\sqrt{a^2-x^2}\,dx&=\int a\cos\theta\,.\,a\cos\theta\,d\theta=a^2\int\cos^2\theta\,d\theta\\&=\frac{a^2}{2}\int(1+\cos2\theta)\,d\theta=\frac{a^2}{2}\left[\theta+\frac{\sin2\theta}{2}\right]+c\\&=\frac{a^2}{2}[\theta+\sin\theta\cos\theta]+c\\&=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{2}\cdot\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}+c\\&=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\tfrac{1}{2}x\sqrt{a^2-x^2}+c.\end{aligned}$$

If we put $y=\sqrt{a^2-x^2}$ we have $y^2+x^2=a^2$ which is the equation of a circle with origin at the centre, and if the Integral is taken from $x=0$ to $x=x\,(ON)$ we see that it is made up of the sector $BOP+\triangle\, ONP$ whose area is $\frac{1}{2}a^2\theta+\frac{1}{2}xy$ where $\angle\, POB=\theta$

$$=\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}+\tfrac{1}{2}x\sqrt{a^2-x^2}.$$

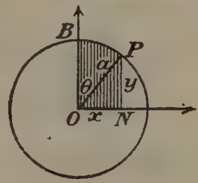

Fig. 156.

If we require $\int_0^a \sqrt{a^2-x^2}\,dx$ we have as limits for θ, $\theta=0$ when $x=0$, and $\theta=\frac{\pi}{2}$ when $x=a$.

$$\therefore\ \int_0^a \sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{a^2}{2}\left(\frac{\pi}{2}\right)=\frac{a^2\pi}{4}.$$

This is the area of a quadrant, $\therefore$ area of whole circle $=\pi a^2$.

Example 17.

Evaluate $$\int\frac{dx}{\sqrt{(4+5x-3x^2)}}.$$

We reduce $4+5x-3x^2$ to the form $a^2-(b+x)^2$.

$$4+5x-3x^2=4-3\left(x^2-\frac{5x}{3}\right)=4-3\left[x^2-\frac{5x}{3}+\left(\frac{5}{6}\right)^2\right]+\frac{25}{12}$$

$$=\frac{73}{12}-3\left(x-\frac{5}{6}\right)^2=3\left\{\frac{73}{36}-\left(x-\frac{5}{6}\right)^2\right\}.$$

$$\therefore\ \text{expression}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\frac{73}{36}-\left(x-\frac{5}{6}\right)^2\right\}}},$$

put $$x-\frac{5}{6}=\frac{\sqrt{73}}{6}\sin\theta,\quad \therefore\ dx=\frac{\sqrt{73}}{6}\cos\theta\,d\theta,$$

and the denominator $=\sqrt{\left(\frac{73}{36}-\frac{73}{36}\sin^2\theta\right)}=\sqrt{\left(\frac{73}{36}\cos^2\theta\right)}=\frac{\sqrt{73}}{6}\cos\theta,$

$$\therefore\ \text{expression}=\frac{1}{\sqrt{3}}\int\frac{\frac{\sqrt{73}}{6}\cos\theta\,d\theta}{\frac{\sqrt{73}}{6}\cos\theta}=\frac{1}{\sqrt{3}}\int d\theta=\frac{\theta}{\sqrt{3}}+c.$$

But $\sin\theta=\dfrac{6x-5}{\sqrt{73}}$ or $\theta=\sin^{-1}\left(\dfrac{6x-5}{\sqrt{73}}\right)$,

$$\therefore\ \text{expression}=\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-5}{\sqrt{73}}\right)+c.$$

Example 18.

Evaluate
$$\int \frac{dx}{4+x^2}.$$

Put $x=2\tan\theta$, then $dx=2\sec^2\theta\,d\theta$ and $4+x^2=4+4\tan^2\theta=4\sec^2\theta$.

$$\therefore\ \text{expression}=\int\frac{2\sec^2\theta\,d\theta}{4\sec^2\theta}=\tfrac{1}{2}\int d\theta=\frac{\theta}{2}=\tfrac{1}{2}\tan^{-1}\left(\frac{x}{2}\right).$$

Note. This result could have been obtained more shortly by using

$$\frac{d}{dx}\left[\tan^{-1}\frac{x}{a}\right]=\frac{a}{a^2+x^2}.$$

Example 19.

Find x when $\dfrac{d^2x}{dt^2}+\mu x=0$, given that $\dfrac{dx}{dt}=0$ when $x=a$ and that $x=a$ when $t=0$.

This example is of importance in Harmonic Motion.

Multiply both sides by $2\dfrac{dx}{dt}$, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\mu x\frac{dx}{dt}=0,$$

i.e.
$$\frac{d}{dt}\left(\frac{dx}{dt}\right)^2+\mu\frac{d}{dt}(x^2)=0;$$

$$\therefore\ \left(\frac{dx}{dt}\right)^2+\mu x^2=c.$$

Since $\dfrac{dx}{dt}=0$ when $x=a$, we find $c=\mu a^2$.

$$\therefore\ \frac{dx}{dt}=\pm\sqrt{\mu}\sqrt{a^2-x^2}.$$

Taking the case when the velocity is negative, we have

$$dt=-\frac{dx}{\sqrt{\mu}\sqrt{a^2-x^2}}.$$

By substituting $x=a\cos\theta$ we get

$$t=\frac{1}{\sqrt{\mu}}\theta+b \text{ and } b=0 \text{ for } x=a \text{ and } \theta=0 \text{ when } t=0.$$

$$\therefore\ \sqrt{\mu}\,t=\cos^{-1}\frac{x}{a} \text{ or } x=a\cos(\sqrt{\mu}\,.\,t).$$

EXAMPLES XIV e

Evaluate the following integrals:

1. $\int \cos^2 x\, dx.$
2. $\int \cos^2 3x\, dx.$
3. $\int \sin ax \cos bx\, dx.$
4. $\int \tan^2 x\, dx.$
5. $\int \sin(1-x)\, dx.$
6. $\int_0^{0 \cdot 8} \sin^2 x \cos x\, dx.$
7. $\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt.$
8. $\int \sin^3 x\, dx.$
9. $\int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x\, dx.$
10. $\int \frac{dx}{\sqrt{(16-x^2)}}.$
11. $\int_0^2 \sqrt{(4-x^2)}\, dx.$
12. $\int \frac{dx}{x^2+3}.$
13. $\int_0^a \frac{dx}{a^2+x^2}.$
14. $\int \frac{dx}{x^2+6x+10}.$
15. $\int \frac{dx}{\sqrt{(3+2x-x^2)}}.$
16. $\int \sqrt{(7-3x^2)}\, dx.$
17. $\int_3^4 \sqrt{(25-x^2)}\, dx.$
18. $\int \frac{dx}{\sqrt{(ax-x^2)}}.$
19. $\int \frac{dx}{\sqrt{(6x-x^2-8)}}.$
20. $\int \cot^2(ax+b)\, dx.$
21. $\int_0^1 (1-x^2)^{\frac{3}{2}}\, dx.$

22. If

$$u=\int_0^T \sin\frac{2\pi rx}{T} \cos\frac{2\pi sx}{T}\, dx,$$

$$v=\int_0^T \sin\frac{2\pi rx}{T} \sin\frac{2\pi sx}{T}\, dx,$$

$$w=\int_0^T \cos\frac{2\pi rx}{T} \cos\frac{2\pi sx}{T}\, dx,$$

where r, s are integers, prove that

(i) $u=0$, (ii) $v=w=0$ if $r \neq s$, (iii) $v=w=\frac{T}{2}$ if $r=s$.

23. Draw the curve $y=\sqrt{4-x^2}$ and find the values of $\int_0^1 \sqrt{4-x^2}\, dx$ and $\int_1^2 \sqrt{4-x^2}\, dx.$

Show without using the Calculus that the results should be $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$ and $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}.$

24. Show that the curve $y=\sqrt{x(1-x)}$ is a circle whose centre is at $(\frac{1}{2}, 0)$; hence evaluate $\int_0^{\frac{1}{2}} \sqrt{x(1-x)}\,dx$. Find this area also by integration.

25. Plot the graph of $\frac{1}{1+x^2}$ and find $\int_0^1 \frac{1}{1+x^2}\,dx$. Evaluate this area also by Simpson's Rule and hence find π.

26. A conductor rotates uniformly about a fixed axis while an electric current x flows along it such that $x=k\sin\theta$, where θ is the angle through which the conductor has turned. Find the maximum value of the current and the average value during the time it is positive. If a^2 is the average value of $k^2\sin^2\theta$, a is called the virtual current. Show that it equals about $0{\cdot}71\,k$.

V. Rational Functions of x

(*a*) When the degree of the numerator is equal to or greater than the degree of the denominator.

Divide the numerator by the denominator until the degree of the remainder is less than that of the denominator.

Example 20.
Evaluate
$$\int \frac{x^2-x}{x+1}\,dx.$$

Now
$$\frac{x^2-x}{x+1}=x-2+\frac{2}{x+1},$$
$$\therefore \text{ integral}=\int\left[x-2+\frac{2}{x+1}\right]dx$$
$$=\tfrac{1}{2}x^2-2x+2\log(x+1)+c.$$

(*b*) When the denominator is quadratic and has no real factors.

If the numerator contains x, this must first be removed by the substitution method.

Example 21.
Evaluate
$$\int \frac{5x-2}{x^2+6x+13}\,dx.$$

Now
$$\frac{d}{dx}(x^2+6x+13)=2x+6;$$
we therefore express $5x-2$ in the form
$$5x-2\equiv\tfrac{5}{2}(2x+6)-15-2\equiv\tfrac{5}{2}(2x+6)-17;$$

$$\therefore \text{ expression} = \tfrac{5}{2}\int \frac{2x+6}{x^2+6x+13}\,dx - 17\int \frac{dx}{x^2+6x+13}$$

$$= \tfrac{5}{2}\int \frac{d(x^2+6x+13)}{x^2+6x+13} - 17\int \frac{dx}{(x+3)^2+4}.$$

The first integral $= \frac{5}{2}\log(x^2+6x+13)$.

In the second integral, put $x+3=2\tan\theta$ so that $dx=2\sec^2\theta\,d\theta$.

Then $\displaystyle\int \frac{dx}{(x+3)^2+4} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta+4} = \tfrac{1}{2}\int d\theta = \frac{\theta}{2} = \tfrac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)$;

$$\therefore \text{ expression} = \tfrac{5}{2}\log(x^2+6x+13) - \tfrac{17}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + c.$$

Note. If $ax^2+2bx+c$ has no real factors, it can be expressed in the form

$$a\left(x^2+\frac{2b}{a}x+\frac{b^2}{a^2}\right)+c-\frac{b^2}{a},$$

or
$$a\left\{\left(x+\frac{b}{a}\right)^2+\frac{ac-b^2}{a^2}\right\},$$

or
$$a\{(x+p)^2+q^2\};$$

and $\displaystyle\int \frac{dx}{(x+p)^2+q^2}$ is evaluated by putting $x+p=q\tan\theta$, its value is

$$\frac{1}{q}\tan^{-1}\left(\frac{x+p}{q}\right).$$

Thus

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c.$$

(*c*) When the denominator is a product of distinct linear factors.

Split up the expression into partial fractions.

Example 22.

Evaluate $$\int \frac{2x-7}{(x-2)(x-3)}\,dx.$$

Let $\displaystyle\frac{2x-7}{(x-2)(x-3)} \equiv \frac{A}{x-2}+\frac{B}{x-3}$, where A, B are constants;

$$\therefore\ 2x-7 \equiv A(x-3)+B(x-2);$$

this is true for *all* values of x.

$\therefore$ when $x=2$, $\quad 4-7=-A$ or $A=3$;

when $x=3$, $\quad 6-7=B$ or $B=-1$.

$$\therefore \text{ expression} = \int\left(\frac{3}{x-2}-\frac{1}{x-3}\right)dx$$
$$=3\log(x-2)-\log(x-3)+c$$
$$=\log\frac{(x-2)^3}{x-3}+c \text{ or } \log\left\{\frac{a(x-2)^3}{x-3}\right\}.$$

Note.

$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)dx=\frac{1}{2a}\Big[-\log(a-x)+\log(a+x)\Big]+c$$
$$=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c.$$

(*d*) When the denominator contains repeated linear factors. As before, split up the expression into partial fractions.

Example 23.

Evaluate $$\int\frac{x-5}{(x+2)(x+3)^2}\,dx.$$

$\dfrac{x-5}{(x+2)(x+3)^2}$ can be expressed in the form $\dfrac{A}{x+2}+\dfrac{qx+r}{(x+3)^2}$.

Since the denominator $(x+3)^2$ is of the second degree, the numerator could contain x, and $\therefore$ the term qx cannot be omitted.

Now

$$\frac{qx+r}{(x+3)^2}=\frac{q(x+3)+r-3q}{(x+3)^2}=\frac{q(x+3)}{(x+3)^2}+\frac{r-3q}{(x+3)^2}=\frac{q}{x+3}+\frac{r-3q}{(x+3)^2}.$$

$\therefore$ $\dfrac{qx+r}{(x+3)^2}$ can be written in the form $\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$.

$\therefore$ $\dfrac{x-5}{(x+2)(x+3)^2}$ can be put $\equiv\dfrac{A}{x+2}+\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$ where A, B, C are constants.

$\therefore$ $x-5\equiv A(x+3)^2+B(x+2)(x+3)+C(x+2)$ for *all* values of x.

$\therefore$ when $x=-3$, $\quad -3-5=-C$ or $C=8$,

when $x=-2$, $\quad -2-5=A$ or $A=-7$,

equate coefficients of x^2,

$$\therefore 0=A+B \text{ or } B=-A=7;$$

$$\therefore \text{ expression} = \int\left[\frac{-7}{x+2}+\frac{7}{x+3}+\frac{8}{(x+3)^2}\right]dx$$
$$= -7\log(x+2)+7\log(x+3)-\frac{8}{x+3}+c$$
$$=\log\left(\frac{x+3}{x+2}\right)^7-\frac{8}{x+3}+c.$$

(*e*) When the denominator contains both linear and irreducible quadratic factors.

Split up the expression into partial fractions and proceed as before.

Example 24.

Evaluate $$\int\frac{3x^2+14}{(x+2)(x^2-2x+5)}\,dx.$$

Let $$\frac{3x^2+14}{(x+2)(x^2-2x+5)}\equiv\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+5},$$
$$\therefore\ 3x^2+14\equiv A(x^2-2x+5)+(Bx+C)(x+2),$$

put $x=-2$, $\therefore\ 12+14=13A$ or $A=2$,

put $x=0$, $\therefore\ 14=5A+2C$ or $2C=14-10$, $C=2$,

equate coefficients of x^2, $\therefore\ 3=A+B$ or $B=1$;

$$\therefore \text{ expression}=\int\left(\frac{2}{x+2}+\frac{x+2}{x^2-2x+5}\right)dx.$$

Now $\dfrac{d}{dx}(x^2-2x+5)=2x-2$, $\therefore$ write $x+2=\frac{1}{2}(2x-2)+3$;

$$\therefore \text{ expression}=2\int\frac{dx}{x+2}+\frac{1}{2}\int\frac{2x-2}{x^2-2x+5}\,dx+3\int\frac{dx}{(x-1)^2+4}$$
$$=2\log(x+2)+\tfrac{1}{2}\log(x^2-2x+5)+\tfrac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right).$$

EXAMPLES XIV f

Evaluate:

1. $\displaystyle\int\frac{x}{1+x}\,dx.$
2. $\displaystyle\int\frac{x+2}{x+3}\,dx.$
3. $\displaystyle\int\frac{x^2+3}{x+2}\,dx.$
4. $\displaystyle\int\frac{dx}{x^2-1}.$
5. $\displaystyle\int\frac{dx}{4x^2-9}.$
6. $\displaystyle\int\frac{x\,dx}{x^2-4}.$
7. $\displaystyle\int\frac{dx}{4x^2+1}.$
8. $\displaystyle\int\frac{3x+2}{4x^2+1}\,dx.$
9. $\displaystyle\int\frac{1-x}{x^2+16}\,dx.$
10. $\displaystyle\int\frac{dx}{x^2-2x+10}.$
11. $\displaystyle\int\frac{5x}{x^2-2x+10}\,dx.$
12. $\displaystyle\int\frac{x+1}{(x+a)^2+b^2}\,dx.$

13. $\displaystyle\int \frac{dx}{(x+1)(x+2)}$. **14.** $\displaystyle\int \frac{x+1}{(x-2)(x-3)}\,dx$.

15. $\displaystyle\int \frac{1+5x}{(1-x)(2+x)}\,dx$. **16.** $\displaystyle\int \frac{1+3x}{10-3x-x^2}\,dx$. **17.** $\displaystyle\int \frac{dx}{x^2(x-1)}$.

18. $\displaystyle\int \frac{5-2x}{(x-1)^2(x+2)}\,dx$. **19.** $\displaystyle\int \frac{2x^3+3x^2+1}{x+2}\,dx$.

20. $\displaystyle\int \frac{30x+51}{(x-2)(x^2+8x+17)}\,dx$. **21.** $\displaystyle\int \frac{dx}{(x-2)(x^2+3x+1)}$.

22. $\displaystyle\int \frac{(x-4)\,dx}{(x-1)(x-2)(x-3)}$. **23.** $\displaystyle\int \frac{x^2\,dx}{(x+1)^3}$.

24. $\displaystyle\int \frac{dx}{x^3+x^2+x}$.

25. Find the area between the x-axis, the curve $y=\dfrac{x}{x+1}$ and the ordinates $x=0$ and $x=1$.

26. Evaluate $\displaystyle\int_0^1 \frac{dx}{x^2+x+1}$.

27. Evaluate $\displaystyle\int_3^4 \frac{x^2\,dx}{x-2}$.

28. Evaluate $\displaystyle\int_1^a \frac{dx}{x(x+1)^2}$, and find its limit as $a \to \infty$.

29. Find the equation of the curve whose gradient at the point (x, y) is $\dfrac{x}{x+2}$, given that the point $(-1, 0)$ lies on the curve.

30. If a motor-car is driven by a constant force and if the air resistance varies as the square of the speed v, then $\dfrac{dv}{dt}=k(V^2-v^2)$, where k, V are constants; find the time taken starting from rest to acquire speed $\frac{1}{2}V$.

VI. Integration by Parts

Since $d(uv)=u\,dv+v\,du$, we have by integrating both sides $uv=\int u\,dv+\int v\,du$.

$$\therefore \int u\,dv = uv - \int v\,du.$$

The success of this method depends upon the possibility of finding $\int v\,du$ instead of $\int u\,dv$.

Example 25.

$$\int x \cos x\, dx = \int x\, d(\sin x) \qquad \text{i.e. } x=u \text{ and } \sin x=v$$

$$= x \sin x - \int \sin x\, dx$$

$$= x \sin x + \cos x + c.$$

This might have been written $\int \cos x\, d\left(\frac{x^2}{2}\right)$ so that $u=\cos x$ and $v=\frac{x^2}{2}$.

Then $$\int \cos x\, d\left(\frac{x^2}{2}\right) = \cos x \frac{x^2}{2} - \int \frac{x^2}{2}\, d(\cos x)$$

$$= \frac{x^2 \cos x}{2} + \tfrac{1}{2}\int x^2 \sin x\, dx.$$

But it is no easier to find $\int x^2 \sin x\, dx$ than to find the original integral, so that it is necessary to make a trial to find which arrangement will be successful.

Example 26. $$\int \log x\, dx = (\log x)(x) - \int x \frac{1}{x}\, dx$$

$$= x \log x - x + c.$$

Example 27. $$z = \int e^{ax} \sin bx\, dx = \frac{1}{a}\int \sin bx\, d(e^{ax})$$

$$= \frac{1}{a} \sin bx\, e^{ax} - \frac{b}{a}\int e^{ax} \cos bx\, dx.$$

But $$\int e^{ax} \cos bx\, dx = \frac{1}{a}\int \cos bx\, d(e^{ax})$$

$$= \frac{1}{a} \cos bx\, e^{ax} + \frac{b}{a}\int e^{ax} \sin bx\, dx.$$

$$\therefore\ z = \frac{1}{a} \sin bx\, e^{ax} - \frac{b}{a}\left[\frac{1}{a} \cos bx\, e^{ax} + \frac{b}{a}\int e^{ax} \sin bx\, dx\right],$$

$$\therefore\ z = \frac{1}{a} \sin bx\, e^{ax} - \frac{b}{a^2} \cos bx\, e^{ax} - \frac{b^2}{a^2} z,$$

$$\therefore\ z\left(1+\frac{b^2}{a^2}\right) = \frac{e^{ax}[a \sin bx - b \cos bx]}{a^2},$$

$$\therefore\ z = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}.$$

EXAMPLES XIV g

Integrate the expressions in Examples 1—18:

1. $x \log x$. **2.** $x e^x$. **3.** $x^2 \log x$.

4. $x \sin x$. **5.** $x(1+x)^{10}$. **6.** $x \cos 3x$.

7. $x \sec^2 x$. **8.** $x \tan^{-1} x$. **9.** $x \sin x \cos x$.

10. $\sin^{-1} x$. **11.** $x^2 \cos x$. **12.** $\dfrac{\log x}{x^3}$.

13. $e^x \sin x$. **14.** $e^x \cos x$. **15.** $\sin^5 x$, $[-\sin^4 x \, d(\cos x)]$

16. $x^n \log x$. **17.** $e^{-2x} \sin 3x$. **18.** $x^2(1-x)^{20}$.

19. Show that

$$n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx.$$

This is called a formula of reduction.

20. Prove $\displaystyle\int x^m \sin x\,dx = -x^m \cos x + m\int x^{m-1} \cos x\,dx.$

Hence find $\displaystyle\int x^5 \sin x\,dx.$

21. Use Example 19 to prove that

$$n\int_0^{\frac{\pi}{2}} \sin^n x\,dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx,$$

and hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 x\,dx.$

22. Obtain a formula of reduction for $\int \cos^n x\,dx$ similar to the result in Example 19; and prove that

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx.$$

23. Obtain a formula of reduction for $\int x^n e^{-x}\,dx$ and hence evaluate

$$\int_0^{\infty} x^{10} e^{-x}\,dx.$$

24. Prove that

$$\int \sin^m \theta \cos^n \theta\,d\theta = -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta \cos^n \theta\,d\theta.$$

Hence, using also Example 22, evaluate

$$\int_0^{\frac{\pi}{2}} \sin^6 \theta \cos^4 \theta\,d\theta \text{ and } \int_0^{\frac{\pi}{2}} \sin^7 \theta \cos^4 \theta\,d\theta.$$

VII. Additional Integrals

(i) Any rational function of $\sin\theta$ and $\cos\theta$ can be expressed as a rational function of $t \equiv \tan\frac{\theta}{2}$,

for $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}} = \frac{2t}{1+t^2},$$

and $$\cos\theta = \frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2},$$

and $dt = d\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1}{2}(1+t^2)\,d\theta, \quad \therefore\ d\theta = \frac{2dt}{1+t^2}.$

Consequently any rational function of $\sin\theta$ and $\cos\theta$ can be reduced to the type dealt with in V.

The following special cases are of importance:

(*a*) $$\int\frac{d\theta}{\sin\theta} = \int\frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \int\frac{\sec^2\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right)}{\tan\frac{\theta}{2}}$$
$$= \int\frac{dt}{t} \qquad \text{where } t\equiv\tan\frac{\theta}{2}$$
$$= \log t = \log\left(\tan\frac{\theta}{2}\right) + c.$$

(*b*) $$\int\frac{d\theta}{\cos\theta} = \int\frac{d\theta}{\sin\left(\frac{\pi}{2}+\theta\right)} = \int\frac{d\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\frac{\pi}{2}+\theta\right)}$$
$$= \log\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\} + c \qquad \text{from } (a).$$

(*c*) $\int\frac{d\theta}{a+b\cos\theta}$, put $t = \tan\frac{\theta}{2}$ and proceed as above.

$$\therefore\ \text{expression} = \int\frac{\frac{2dt}{1+t^2}}{a+\frac{b(1-t^2)}{1+t^2}} = 2\int\frac{dt}{(a+b)+(a-b)t^2}.$$

Suppose $a > b$, then it

$$= \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b} + t^2}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left\{t\sqrt{\left(\frac{a-b}{a+b}\right)}\right\} + c$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\tan\frac{\theta}{2}\sqrt{\left(\frac{a-b}{a+b}\right)}\right\} + c.$$

If $a < b$, the integral involves a logarithm and is effected by partial fractions.

(ii) Functions involving $\sqrt{(x^2+a^2)}$ or $\sqrt{(x^2-a^2)}$ are best treated by hyperbolic substitutions, see p. 272; but the following methods are sometimes used.

(*a*) $$\int \frac{dx}{\sqrt{(x^2+a^2)}}.$$

Put $z - x = \sqrt{(x^2+a^2)}$;

$$\therefore\ (z-x)^2 = x^2 + a^2 \text{ or } z^2 - 2zx + x^2 = x^2 + a^2,$$

$$\therefore\ z^2 - 2zx = a^2,$$

$$\therefore\ 2z\,dz - 2(z\,dx + x\,dz) = 0,$$

$$\therefore\ dz\,(z-x) = z\,dx,$$

$$\therefore\ \frac{dz}{z} = \frac{dx}{z-x},$$

$$\therefore\ \int\frac{dx}{\sqrt{(x^2+a^2)}} = \int\frac{dx}{z-x} = \int\frac{dz}{z} = \log z + c$$

$$= \log\left[x + \sqrt{(x^2+a^2)}\right] + c.$$

(*b*) $$\int\sqrt{(x^2+a^2)}\,dx.$$

By parts, $$\int\sqrt{(x^2+a^2)}\,dx = x\sqrt{(x^2+a^2)} - \int\frac{x^2\,dx}{\sqrt{(x^2+a^2)}}.$$

But

$$\int\sqrt{(x^2+a^2)}\,dx = \int\frac{x^2+a^2}{\sqrt{(x^2+a^2)}}\,dx = \int\frac{x^2\,dx}{\sqrt{(x^2+a^2)}} + a^2\int\frac{dx}{\sqrt{(x^2+a^2)}};$$

$\therefore$ adding $2\int\sqrt{(x^2+a^2)}\,dx = x\sqrt{(x^2+a^2)} + a^2\int\frac{dx}{\sqrt{(x^2+a^2)}}$,

$$\therefore \int\sqrt{(x^2+a^2)}\,dx = \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{a^2}{2}\log\left[x+\sqrt{(x^2+a^2)}\right] + c.$$

(iii) There are certain integrals which for special limits can be easily calculated; we shall discuss one important definite integral of this kind.

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta.$$

For brevity, write $\sin\theta \equiv s$ and $\cos\theta \equiv c$.

$$\begin{aligned}\text{Now}\quad \frac{d}{d\theta}(s^{m+1}c^{n-1}) &= (m+1)\,s^m c^n - (n-1)\,s^{m+2}c^{n-2}\\ &= (m+1)\,s^m c^n - (n-1)\,s^m c^{n-2}(1-c^2)\\ &= (m+n)\,s^m c^n - (n-1)\,s^m c^{n-2}.\end{aligned}$$

Integrate this from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Since $s = 0$ if $\theta = 0$ and $c = 0$ if $\theta = \frac{\pi}{2}$, $\left[s^{m+1}c^{n-1}\right]_0^{\frac{\pi}{2}} = 0$,

$$\therefore \int_0^{\frac{\pi}{2}} s^m c^n\,d\theta = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}} s^m c^{n-2}\,d\theta \quad\text{.....................(1)}$$

$$= \text{by symmetry}\ \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} s^{m-2}c^n\,d\theta \quad\text{...(2).}$$

Put $m = 0$ in (1), then

$$\int_0^{\frac{\pi}{2}} c^n\,d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} c^{n-2}\,d\theta \quad\text{...............(3).}$$

Put $n = 0$ in (2), then

$$\int_0^{\frac{\pi}{2}} s^m\,d\theta = \frac{m-1}{m}\int_0^{\frac{\pi}{2}} s^{m-2}\,d\theta \quad\text{...............(4).}$$

The results in (1), (2), (3), (4) enable us to evaluate the integral in any case: we shall illustrate the various possibilities by examples.

Example 28.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^8\theta\, d\theta \text{ and } \int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^8\theta\, d\theta = \frac{7}{8}\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{7}{8}\times\frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}.$$

$\int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta$ has the same value.

Example 29.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta = \frac{6}{7}\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta$$

$$= \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}.$$

Note the difference in form of the answer according as the index is even or odd.

Example 30.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta = \frac{3}{10}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^6\theta\, d\theta = \frac{3}{10}\times\frac{1}{8}\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$$

$$= \frac{3}{10}\times\frac{1}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{3}{10}\times\frac{1}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}.$$

Example 31.

Evaluate $$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta\, d\theta.$$

If either index is *odd*, reduce that one.

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^5\theta\, d\theta = \frac{4}{9}\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^3\theta\, d\theta = \frac{4}{9}\times\frac{2}{7}\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta$$

$$= \frac{4}{9}\times\frac{2}{7}\int_0^{\frac{\pi}{2}} \sin^4\theta\, d(\sin\theta) = \frac{4}{9}\times\frac{2}{7}\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{4}{9}\times\frac{2}{7}\times\frac{1}{5}.$$

Example 32.

Evaluate $$\int_0^1 x^2(1-x^2)^{\frac{3}{2}}\, dx.$$

Put $x=\sin\theta$, then expression

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\, d\theta$$

$$= \frac{1}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{32}.$$

EXAMPLES XIV h

Evaluate

1. $\int \frac{d\theta}{\sin\frac{\theta}{2}}$.
2. $\int \frac{d\theta}{\cos\frac{\theta}{2}}$.
3. $\int \frac{d\theta}{\sin\theta\cos\theta}$.
4. $\int \frac{dx}{\sin x+\cos x}$.
5. $\int \frac{dx}{5+3\cos x}$.
6. $\int \frac{dx}{3+5\cos x}$.
7. $\int_1^2 \frac{dx}{\sqrt{(1+x^2)}}$.
8. $\int_0^1 \sqrt{(1+x^2)}\, dx$.
9. $\int_0^\pi \frac{dx}{3+2\cos x}$.
10. $\int (1+x^2)^{\frac{3}{2}}\, dx$.
11. $\int \sqrt{(x^2-16)}\, dx$.
12. $\int \sqrt{(x^2-3x-10)}\, dx$.
13. $\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta$.
14. $\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$.
15. $\int_0^{\frac{\pi}{2}} \sin^{10}\theta\cos^3\theta\, d\theta$.
16. $\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\, d\theta$.
17. $\int_0^1 (1-x^2)^{\frac{5}{2}}\, dx$.
18. $\int_0^1 x^4(1-x^2)^3\, dx$.
19. $\int_0^1 x^3(1-x)^4\, dx$.
20. $\int_0^a x^2\sqrt{(a^2-x^2)}\, dx$.
21. $\int_0^2 x\sqrt{(2x-x^2)}\, dx$.

22. Find the area enclosed by the curve $y^2=x^3(1-x)$.

23. Sketch the curve given by $x=a\sin^3\theta$, $y=a\sin^2\theta\cos\theta$, and find the area of the portion for which x is positive.

24. Evaluate $\int_0^1 \frac{x^4\,dx}{\sqrt{(1-x^2)}}$.

25. Find the area enclosed by the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$.

EXAMPLES XIV i

Miscellaneous Examples on Integration

Find the integrals of the expressions in Examples 1—50:

1. $x^{-1.37}$. **2.** $\sqrt[3]{(a+x)}$. **3.** $\frac{1}{1+x^2}$.

4. $\frac{x}{(1+2x)^{\frac{1}{3}}}$. **5.** $\frac{x}{\sqrt{(a^2-x^2)}}$. **6.** $\frac{1}{c-x}$.

7. $\frac{x^2}{1-2x^3}$. **8.** $\frac{x+a}{x^2+a^2}$. **9.** $\frac{\sin x}{a+b\cos x}$.

10. $\frac{1}{(a+bx)^2}$. **11.** $\frac{x}{(1-3x)^2}$. **12.** $\frac{1}{x^2(1+x)}$.

13. $\frac{x}{\sqrt{(2+3x)}}$. **14.** $\frac{\sqrt{(1+\log x)}}{x}$. **15.** $\frac{1}{x\log a}$.

16. $\sin^2 x\cos^2 x$. **17.** $\cot x$. **18.** $\operatorname{cosec} 3x$.

19. $\operatorname{cosec}^2 3x$. **20.** $\frac{x^3}{(2+x)^2}$. **21.** $\sin ax\sin bx$.

22. $\frac{5x^3+1}{(x-1)(x-2)}$. **23.** $\frac{x}{x^2+2x-3}$. **24.** $\frac{x}{(1+x)(1+x^2)}$.

25. $\sin^2 3x$. **26.** $x\log x$. **27.** $\cos^{-1}x$.

28. $\sec x$. **29.** $\sin^3 2x\cos 2x$. **30.** $\frac{\sqrt{(x^2-a^2)}}{x}$.

31. $x(3-2x)^{\frac{3}{2}}$. **32.** $\frac{x^2-5}{2\sqrt{x}}$. **33.** $\frac{\log(ax)}{x}$.

34. $\frac{\cos x}{1+\sin x}$. **35.** x^2e^x. **36.** $\frac{\sin(\log x)}{x}$.

37. $\frac{\log(x^2)}{x}$. **38.** $\frac{1}{x(1+\log x)}$. **39.** $\frac{\tan^{-1}x}{1+x^2}$.

40. $\sqrt{(e^x+4)}$. **41.** $\tan^2(ax+b)$. **42.** 7^x.

43. $\log_2 x$. **44.** $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$. **45.** $\dfrac{x}{\sqrt{(x-1)}}$.

46. $\tan^3 x \sec^2 x$. **47.** $e^{-x}\sin 2x$. **48.** $\tan^3 x$.

49. $\dfrac{1}{\cos^2 x + 4\sin^2 x}$. **50.** $\dfrac{1}{x\sqrt{(a^2-x^2)}}$.

51. Prove that

$$\int_0^x \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int_0^x \tan^{n-2} x\,dx,$$

and evaluate

$$\int_0^{\frac{\pi}{4}} \tan^4 x\,dx.$$

Evaluate the definite integrals in Examples 52—63:

52. $\displaystyle\int_0^\infty e^{-2x}\,dx$. **53.** $\displaystyle\int_0^1 x^2(1-x)^3\,dx$. **54.** $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta$.

55. $\displaystyle\int_0^\pi \sin^2 10x\,dx$. **56.** $\displaystyle\int_0^{\frac{\pi}{4}} x\sin x\,dx$. **57.** $\displaystyle\int_0^\infty x^2 e^{-3x}\,dx$.

58. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$. **59.** $\displaystyle\int_0^1 (1-x^2)^{\frac{7}{2}}\,dx$. **60.** $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{(1-x^2)}}$.

61. $\displaystyle\int_0^1 \frac{x+1}{(x+2)(x+3)}\,dx$. **62.** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\theta \operatorname{cosec}\theta\,d\theta$.

63. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\cos x\,dx$.

64. Two long coaxal cylinders of radii a and b are charged to form a condenser. If the potential of the outer one is zero, then the potential V of the inner is given by

$$V = 2\int_0^\infty \left[\frac{2\pi a\sigma\,dz}{\sqrt{(a^2+z^2)}} - \frac{2\pi a\sigma\,dz}{\sqrt{(b^2+z^2)}}\right]$$

where σ is a constant: prove that

$$V = 4\pi a\sigma \log\left(\frac{b}{a}\right).$$

65. Prove that

$$\int_0^1 x \log\left(1+\tfrac{1}{2}x\right) dx = \tfrac{3}{4}\left(1-2\log\tfrac{3}{2}\right).$$

66. Prove that $\displaystyle\int_2^{\frac{5}{2}} \sqrt{(x^2-4)}\, dx = \tfrac{15}{8} - 2\log 2.$

67. Evaluate $$\int_0^{\frac{\pi}{2}} \cos^2 x \cos 2x\, dx.$$

68. Find the area of the loop of the curve $y^2 = x(x-1)^2$.

69. The curve $y = 1 + \sin x$ is rotated about the x-axis; prove that the volume contained between the surface and the planes $x=0$ and $x=\pi$ is

$$\pi\left(4+\frac{3\pi}{2}\right).$$

70. Prove that the area of the curve $x^4 - x^2 + y^2 = 0$ is $\frac{4}{3}$.

y	$\int y\,dx$	PAGE
$\tan x$	$\log(\sec x)+c$	215
$\operatorname{cosec} x$	$\log\left(\tan\frac{x}{2}\right)+c$	227
$\sec x$	$\log\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}+c$	227
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right)+c$	222
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$	218
$\dfrac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)+c$	216
$\dfrac{1}{\sqrt{(a^2+x^2)}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)+c$	272
	or $\log[x+\sqrt{(a^2+x^2)}]+c$	228
$\dfrac{1}{\sqrt{(x^2-a^2)}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)+c$	273, no. 4

CHAPTER XV

APPLICATIONS TO GEOMETRY

I. The Tangent to a curve in Polar Coordinates

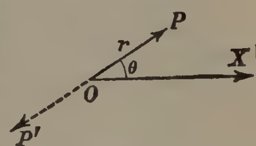

Fig. 157.

IF O is a fixed point in a fixed straight line OX, called the *initial line*, the position of any point P in a fixed plane through OX is determined when the length r of OP and the angle θ which OP makes with OX are given: (r, θ) are called the *polar coordinates* of P; r is called the *radius vector* and θ is called the *vectorial angle.*

It is usual to measure θ as positive for an anti-clockwise rotation from OX and negative for a clockwise rotation: a negative value of r is represented by producing the radius vector backwards through O. Thus if in Fig. 157

$$OP = OP' = 2 \text{ and } \angle XOP = \frac{\pi}{6},$$

the point P could be represented in any of the following ways:

$$\left(2, \frac{\pi}{6}\right), \left(-2, \frac{7\pi}{6}\right), \left(-2, -\frac{5\pi}{6}\right), \left(2, -\frac{11\pi}{6}\right)$$

and P' by $$\left(-2, \frac{\pi}{6}\right), \left(2, -\frac{5\pi}{6}\right) \text{ etc.}$$

If the Cartesian coordinates of P are (x, y), we have

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} r &= \sqrt{(x^2+y^2)} \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\}.$$

The *Polar equation of a curve* is the relation between the polar coordinates r, θ of any point on the curve.

Example 1.

Find the polar equation of a circle if the origin lies on the circumference and the centre lies on the initial line.

Let the radius $=a$, $\therefore OA=2a$, $OP=r$, $\angle POA=\theta$.

Since $\angle OPA=\frac{\pi}{2}$, $\underline{r=2a\cos\theta}$: this is the required equation.

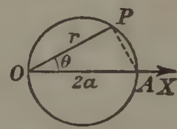

Fig. 158.

Note. When $\pi>\theta>\frac{\pi}{2}$, $\cos\theta$ is negative and $\therefore r$ is negative. The radius vector is $\therefore$ produced backwards to meet the circle at P': see Fig. 159. The whole circumference is described by the variation of θ from 0 to π.

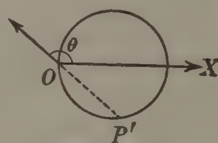

Fig. 159.

Further Notation.

If the tangent at any point $P(r, \theta)$ on the curve $r=f(\theta)$ meets OX at T (Fig. 160) and if OY is the perpendicular from O to PT and if A is any fixed point on the curve,

$\angle OPT$ is denoted by ϕ,

OY is denoted by p,

the length of arc AP is denoted by s,

$\angle PTX$ is denoted by ψ.

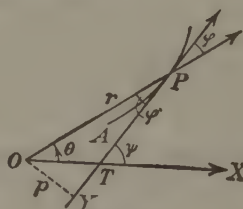

Fig. 160.

We shall now establish the following results:

$$\textbf{(i)}\ \sin\phi=r\frac{d\theta}{ds};\quad \textbf{(ii)}\ \cos\phi=\frac{dr}{ds};$$

$$\textbf{(iii)}\ \tan\phi=r\frac{d\theta}{dr};\quad \textbf{(iv)}\ p=r\sin\phi.$$

In Fig. 161 take the point $Q(r+\delta r, \theta+\delta\theta)$ near P and draw PN perpendicular to OQ.

Since arc $AP=s$, arc $AQ=s+\delta s$ and arc $PQ=\delta s$.

Now $\angle PON=\delta\theta$,

$$\therefore PN=r\sin\delta\theta \triangleq r\,\delta\theta.$$

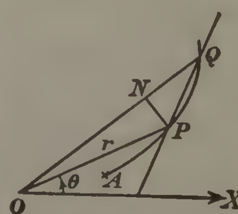

Fig. 161.

Also $$NQ = OQ - ON = (r + \delta r) - r\cos\delta\theta$$
$$= \delta r + r(1-\cos\delta\theta) = \delta r + 2r\sin^2\frac{\delta\theta}{2}$$
$$\doteq \delta r + 2r \times \frac{(\delta\theta)^2}{4} \doteq \delta r + \tfrac{1}{2}r(\delta\theta)^2 \doteq \delta r.$$

Also chord $PQ \doteq$ arc $PQ \doteq \delta s$.

$$\therefore \quad \sin NQP \doteq \frac{r\delta\theta}{\delta s}, \quad \cos NQP \doteq \frac{\delta r}{\delta s}, \quad \tan NQP \doteq \frac{r\delta\theta}{\delta r}.$$

In the limit when $\delta\theta \to 0$, $\angle NQP \to \phi$ and we have

$$\sin\phi = r\frac{d\theta}{ds}, \quad \cos\phi = \frac{dr}{ds}, \quad \tan\phi = r\frac{d\theta}{dr}.$$

Also since $\angle OYP = \frac{\pi}{2}$, we have $p = r\sin\phi$.

EXAMPLES XV a

1. Find ϕ and p in terms of r, θ for the following loci:

(i) $r = a\sin\theta$; (ii) $r\sin\theta = a$;

(iii) $r(1+\sin\theta) = a$; (iv) $r = e^{\theta\cot\alpha}$.

2. Prove that for the curve $r^2 = a^2\cos 2\theta$, the relation between p, r is $r^3 = a^2 p$. [The p, r equation is called the *pedal equation* of the curve.]

3. Find the pedal equation of the parabola $\frac{2a}{r} = 1 + \cos\theta$.

4. Find the pedal equation of the curve $r^n = a^n\sin n\theta$.

5. Find a point on the curve $r^2 = a^2\cos 2\theta$ at which the tangent makes an angle $\frac{3\pi}{4}$ with the initial lines.

6. Sketch $r = a\theta$, the spiral of Archimedes, and find for what value of r, $\phi = \frac{\pi}{4}$.

7. Prove that for any curve $\frac{ds}{d\theta} = \frac{r^2}{p}$.

8. Prove that for any curve $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

II. Areas in polar coordinates

The area bounded by the curve $r=f(\theta)$ and the lines $\theta=\alpha$, $\theta=\beta$ is $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$.

With the notation of Fig. 161, p. 236, let the area of the sector AOP be z.

Then $\delta z =$ area of sector POQ.

$$\therefore\ \tfrac{1}{2}r^2\sin\delta\theta < \delta z < \tfrac{1}{2}(r+\delta r)^2\sin\delta\theta,$$

$$\therefore\ \tfrac{1}{2}r^2\frac{\sin\delta\theta}{\delta\theta} < \frac{\delta z}{\delta\theta} < \tfrac{1}{2}(r+\delta r)^2\frac{\sin\delta\theta}{\delta\theta}.$$

$\therefore$ when $\delta\theta\to 0$ since $\delta r\to 0$ and $\dfrac{\sin\delta\theta}{\delta\theta}\to 1$, we have

$$\frac{dz}{d\theta}=\tfrac{1}{2}r^2.$$

$$\therefore\ z=\tfrac{1}{2}\int r^2\,d\theta + c.$$

But $z=0$ when $\theta=\alpha$.

$\therefore$ area of sector $AOP \quad = \frac{1}{2}\int_\alpha^\theta r^2\,d\theta$

and the required area $\quad = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta.$

Note. If we use the formula, the area of $\Delta\ OPQ=\frac{1}{2}$ height $\times$ base, we have $\delta z \doteq \frac{1}{2}p\,.\,\delta s$.

If $s=s_1$ and $s=s_2$ correspond to $\theta=\alpha$ and $\theta=\beta$, we then have area of sector $=\frac{1}{2}\int_{s_1}^{s_2} p\,.\,ds$.

Example 2.

Trace the cardioid $r=a(1+\cos\theta)$ and find the area it encloses.

Draw the circle with diameter $OA=a$, join O to any point Q on the circle and produce OQ to P so that $QP=a$, then the locus of P is the required curve, for

$$OP=OQ+QP=a\cos\theta+a.$$

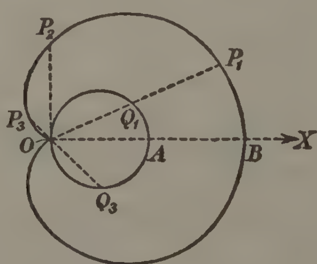

Fig. 162.

When $\theta=\frac{\pi}{2}$, $OP_2=a$; when $\theta=\frac{2\pi}{3}$, $OQ_3=-\frac{a}{2}$, $OP_3=a-\frac{a}{2}=\frac{a}{2}$; when $\theta=\pi$, $OQ=-a$ and $OP=0$ so that P is at O.

The curve is traced out once completely when θ varies from 0 to 2π.

$$\therefore \text{area} = \tfrac{1}{2}\int_0^{2\pi} r^2\,d\theta = \int_0^{\pi} r^2\,d\theta = a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta$$

$$=a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=a^2\int_0^{\pi}\left\{1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right\}d\theta$$

$$=a^2\left[\tfrac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\pi}=\frac{3\pi a^2}{2}.$$

Alternatively $a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta=4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta.$

Putting $\theta=2\omega$ the integral becomes

$$8a^2\int_0^{\frac{\pi}{2}}\cos^4\omega\,d\omega=8a^2\times\tfrac{3}{4}\times\tfrac{1}{2}\times\frac{\pi}{2}=\frac{3\pi a^2}{2}.$$

(See pp. 229—230.)

Example 3.

Prove that the tangent to the cardioid $r=a(1+\cos\theta)$ at the point (r, θ) makes an angle $\frac{\pi}{2}+\frac{\theta}{2}$ with the radius vector.

$$r=a(1+\cos\theta)=2a\cos^2\frac{\theta}{2}.$$

$$\therefore \log r=\log 2a+2\log\cos\frac{\theta}{2}, \quad \therefore \frac{1}{r}\frac{dr}{d\theta}=-\tan\frac{\theta}{2}.$$

But $\tan\phi=r\dfrac{d\theta}{dr}$, $\therefore \cot\phi=\dfrac{1}{r}\dfrac{dr}{d\theta}=-\tan\dfrac{\theta}{2}=\cot\left(\dfrac{\pi}{2}+\dfrac{\theta}{2}\right).$

$$\therefore \phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

EXAMPLES XV b

1. Find the area enclosed by $r=a\cos\theta$.
2. Find the area enclosed by $r=a(1-\cos\theta)$.
3. For what values of θ is r^2 negative if $r^2=a^2\cos 2\theta$?

Sketch the graph of $r^2=a^2\cos 2\theta$ and find the area of one loop.

4. Prove that for $r\theta=a$ the area from (r_1, θ_1) to (r_2, θ_2) is proportional to r_1-r_2.

5. Find the area between the hyperbola $r^2 \sin 2\theta = 2c^2$ and the lines

$$\theta=\frac{\pi}{6}, \quad \theta=\frac{\pi}{3}.$$

6. Trace the curve $r=1+2\cos\theta$ and show that it consists of two loops. What area is obtained by integrating $\frac{1}{2}r^2\,d\theta$ from 0 to 2π? What is the relation between r and θ for the inner loop? What limits of integration for θ lead to the area of the inner loop?

7. Using the relations $r^2=x^2+y^2$, $\tan\theta=\dfrac{y}{x}$,

prove that $$\tfrac{1}{2}\int r^2\,d\theta=\tfrac{1}{2}\int(x\,dy-y\,dx).$$

8. The polar equation of an ellipse with its centre O as origin and its semi-major axis OA as initial line is $r^2\left(\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}\right)=1$; P is a point on the curve such that $\angle AOP=\alpha$; find the area of the sector AOP.

III. The Centroid in Polar Coordinates

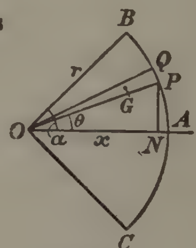

Fig. 163.

Find the centroid of (i) a circular arc, (ii) a circular sector.

(i) If 2α is the angle subtended by the arc at the centre, take the bisector as the initial line. Then if $AP=s$,

$$PQ=\delta s=r\delta\theta, \quad x=ON$$

and $$\text{arc } CAB=2r\,.\,\alpha.$$

Then $$\bar{x} \triangleq \frac{\Sigma\,\delta s\,.\,x}{\Sigma\,\delta s}=\frac{\int_{-\alpha}^{+\alpha}(r\,d\theta)(r\cos\theta)}{\int_{-\alpha}^{+\alpha}r\,d\theta}$$

$$=\frac{r^2\int_{-\alpha}^{+\alpha}\cos\theta\,d\theta}{\int_{-\alpha}^{+\alpha}r\,d\theta}=\frac{r^2\Big[\sin\theta\Big]_{-\alpha}^{+\alpha}}{r\Big[\theta\Big]_{-\alpha}^{+\alpha}}=\frac{r^2\,.\,2\sin\alpha}{r\,.\,2\alpha}$$

$$=\frac{r\sin\alpha}{\alpha}.$$

(ii) The centroid of the element POQ will be at G whose abscissa is approximately $\frac{2}{3} r \cos\theta$;

$$\therefore \bar{x} \doteq \frac{\Sigma\left(\frac{1}{2} r^2 \delta\theta\right)\left(\frac{2}{3} r \cos\theta\right)}{\Sigma\left(\frac{1}{2} r^2 \delta\theta\right)},$$

$$\bar{x} = \frac{\frac{1}{3}\int_{-a}^{+a} r^3 \cos\theta \, d\theta}{\frac{1}{2}\int_{-a}^{+a} r^2 \, d\theta} = \frac{2}{3} r \frac{\sin a}{a}.$$

EXAMPLES XV c

1. Find the centre of gravity of a wire in the form of a semicircular arc.

2. Find the centre of gravity of a semicircular area.

3. P is any point on the semicircle on OA as diameter; the equation of the circle with origin O and initial line OA is $r = 2a\cos\theta$, where a is the radius; show that the distance from OA of the centre of gravity of the area AOP is $\frac{2a}{3} \cdot \frac{1-\cos^4 a}{a + \sin a \cos a}$, where $\angle AOP = a$.

4. Prove that the centre of gravity of the area bounded by the cardioid $r = a(1+\cos\theta)$ lies on the initial line at a distance $\frac{5a}{6}$ from the origin.

5. Find the distance from the origin of the centre of gravity of one loop of $r^2 = a^2 \cos 2\theta$.

6. If O is the origin and if the initial line cuts the cardioid

$$r = a(1+\cos\theta)$$

at A and if $P(r, \theta)$ is any point on the curve, it can be proved that the length of the arc $AP = 4a \sin\frac{\theta}{2}$; use this result to find the centre of gravity of a wire in the form of a curve $r = a(1+\cos\theta)$.

IV. Length of an arc

(i) *Cartesian Coordinates.*

P, Q are the points (x, y), $(x+\delta x, y+\delta y)$ on the curve $y = f(x)$, see Fig. 164.

A is any fixed point on the curve, and the length of arc $\overparen{AP} = s$, $\overparen{PQ} = \delta s$.

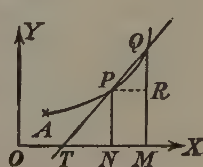

Fig. 164.

Now $\overline{PQ}^2 = PR^2 + RQ^2$, where $\overline{PQ}$ is the chord PQ,

$$\text{i.e. } \overline{PQ}^2 = \delta x^2 + \delta y^2.$$

$$\therefore \left|\frac{\overline{PQ}}{\delta x}\right|^2 = 1 + \left|\frac{\delta y}{\delta x}\right|^2$$

which may be written

$$\left(\frac{\overline{PQ}}{\widehat{PQ}} \times \frac{\widehat{PQ}}{\delta x}\right)^2 = 1 + \left|\frac{\delta y}{\delta x}\right|^2.$$

When $\delta x \to 0$, the limit of $\dfrac{\overline{PQ}}{\widehat{PQ}} = 1$ and $\underset{\delta x \to 0}{\mathrm{Lt}} \dfrac{\delta s}{\delta x} = \dfrac{ds}{dx}$.

$$\therefore \left|\frac{ds}{dx}\right|^2 = 1 + \left|\frac{dy}{dx}\right|^2,$$

$$\text{i.e. } \frac{ds}{dx} = \sqrt{\left\{1 + \left|\frac{dy}{dx}\right|^2\right\}} \text{ and } s = \int \sqrt{\left(1 + \left|\frac{dy}{dx}\right|^2\right)} dx + c.$$

$\therefore$ the length of arc from $(x_1\, y_1)$ to $(x_2\, y_2)$ is

$$\int_{x_1}^{x_2} \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx.$$

For most curves, this integral cannot be expressed in terms of elementary functions.

If the curve is given by $x = f(t)$, $y = \theta(t)$ by using the relation $(\delta s)^2 \triangleq (\delta x)^2 + (\delta y)^2$ we have $\left(\dfrac{ds}{dt}\right)^2 = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$ from which s is found by integration.

If the chord QP meets OX at T (Fig. 164), we have $\cos PTN = \cos QPR = \dfrac{PR}{PQ} = \dfrac{\delta x}{\delta s} \times \dfrac{\delta s}{PQ}$.

If the tangent at P makes an angle ψ with OX (Fig. 165), then

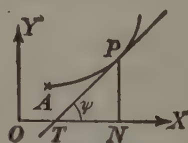

Fig. 165.

$$\cos\psi = \mathrm{Lt}\, \frac{\delta x}{\delta s} \times \frac{\delta s}{\overline{PQ}} = \frac{dx}{ds}.$$

Similarly $\sin\psi = \mathrm{Lt}\, \dfrac{\delta y}{\delta s} \times \dfrac{\delta s}{\overline{PQ}} = \dfrac{dy}{ds}$ and as usual $\tan\psi = \dfrac{dy}{dx}$.

From these results, since $\sec^2\psi = 1 + \tan^2\psi$, we have

$$\overline{\frac{ds}{dx}}\bigg|^2 = 1 + \overline{\frac{dy}{dx}}\bigg|^2$$

as before.

Example 4.

Find the length of the curve $y = \sqrt{a^2 - x^2}$ from $x = 0$ to $x = \frac{a}{2}$.

$$\frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}}, \quad \therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

$$\therefore \frac{ds}{dx} = \frac{a}{\sqrt{(a^2 - x^2)}} \text{ or } s = \int \frac{a\,dx}{\sqrt{(a^2 - x^2)}}.$$

$$\therefore \text{length of arc required} = \int_0^{\frac{a}{2}} \frac{a\,dx}{\sqrt{(a^2 - x^2)}} = a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{\frac{a}{2}}$$

$$= a\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = a \times \frac{\pi}{6} = \frac{\pi a}{6}.$$

Note. The equation $y = \sqrt{(a^2 - x^2)}$ or $y^2 + x^2 = a^2$ represents a circle radius a, centre the origin: and the required result is of course obtained at once from elementary trigonometry.

(ii) *Polar Coordinates.*

With the notation of p. 236, we have

$$\tan\phi = r\frac{d\theta}{dr}, \quad \cos\phi = \frac{dr}{ds}.$$

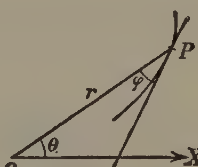

Fig. 166.

But

$$\sec^2\phi = 1 + \tan^2\phi, \quad \therefore \left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2;$$

$$\therefore \frac{ds}{dr} = \sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}} \text{ and } s = \int\sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}}\,dr.$$

This integral cannot usually be expressed in terms of elementary functions: and even when this can be done, it is often better first of all to find ϕ in terms of r, θ and then to integrate either

$$\frac{ds}{dr} = \sec\phi \quad \text{or} \quad \frac{1}{r}\frac{ds}{d\theta} = \operatorname{cosec}\phi.$$

Example 5.

Find the length of the cardioid $r=a(1+\cos\theta)$.

The graph is given on p. 238.

We have $r=2a\cos^2\frac{\theta}{2}$, $\therefore \log r=\log(2a)+2\log\left(\cos\frac{\theta}{2}\right)$.

$$\therefore \frac{1}{r}\frac{dr}{d\theta}=-\tan\frac{\theta}{2} \text{ or } \cot\phi=-\tan\frac{\theta}{2}=\cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right);$$

$$\therefore \phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

$$\therefore \frac{ds}{d\theta}=r\operatorname{cosec}\phi=2a\cos^2\frac{\theta}{2}\operatorname{cosec}\left(\frac{\pi}{2}+\frac{\theta}{2}\right)=2a\cos^2\frac{\theta}{2}\sec\frac{\theta}{2}=2a\cos\frac{\theta}{2};$$

$$\therefore s=\int 2a\cos\frac{\theta}{2}\,d\theta+c$$

$$=4a\sin\frac{\theta}{2}+c.$$

If we measure $s=0$ from $\theta=0$ we have $c=0$;

$$\therefore s=4a\sin\frac{\theta}{2}.$$

$\therefore$ the length of the portion above OX is obtained by putting $\theta=\pi$.

$\therefore$ total length of curve $=2\times 4a\sin\frac{\pi}{2}=8a$.

EXAMPLES XV d

1. Find the length of the arc of $x^3=y^2$ from $x=0$ to $x=1$.
2. For the curve $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, prove that

$$\frac{ds}{d\theta}=2a\cos\frac{\theta}{2}$$

and calculate the length of the portion of the curve (a cycloid) between $\theta=0$ and $\theta=\pi$.

3. Find the total length of the curve $r=a(1-\cos\theta)$.
4. Find the length of the arc of $x^3=8y^2$ from $x=0$ to $x=2$.
5. Find the length of the curve $x=a\cos^3\theta$, $y=a\sin^3\theta$.
6. Find the length of the catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$ from $x=0$ to $x=a$.
7. Show that for the parabola $y^2=4ax$, $\frac{ds}{dx}=\sqrt{\left(1+\frac{a}{x}\right)}$.

8. Show that for the parabola $y=ax^2$, $\frac{ds}{dx}=\sqrt{(1+4a^2x^2)}$: if x is small, $\frac{ds}{dx} \simeq 1+2a^2x^2$; use this result to find the length of arc from (0, 0) to (h, k) if h is small.

9. Find the length of arc cut off between $\theta=\alpha$ and $\theta=\beta$ from $r=a \cdot e^{\theta \cot \gamma}$.

10. For the curve $r \cos^3 \theta = a \sin^2 \theta$, prove that

$$\frac{ds}{d\theta}=a \tan \theta \sec^2 \theta \,(4+9 \tan^2 \theta)^{\frac{1}{2}}$$

and show that the length of the portion of the curve between $\theta=0$ and $\theta=\tan^{-1}\left(\frac{\sqrt{5}}{3}\right)$ is $\frac{19a}{27}$.

V. Curvature

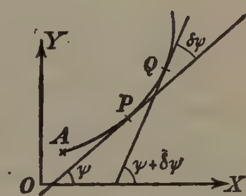

Fig. 167.

A is a fixed point on the curve $y=f(x)$; the tangents at

$P\,(x, y)$ and $Q\,(x+\delta x, y+\delta y)$

make angles ψ, $\psi+\delta\psi$ with OX; and the arcs AP, AQ are of lengths s, $s+\delta s$.

The angle between the tangents at P, Q is $\delta\psi$ and arc $PQ=\delta s$.

$\therefore \frac{\delta\psi}{\delta s}$ measures the average change in direction per unit increase of arc for the arc PQ (i.e. the rate at which the curve bends from P to Q) and is called the *average curvature* of the arc PQ.

When $\delta s \to 0$, Lt $\frac{\delta\psi}{\delta s}=\frac{d\psi}{ds}$ and this is called the *curvature* of the curve at P.

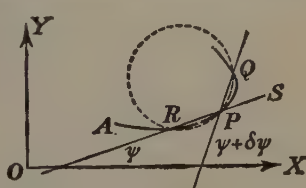

Fig. 168.

Consider three points P, Q, R on the curve (Fig. 168) given by

arc $AP=s$, arc $AQ=s+\delta s$,

arc $AR=s-\delta s$

and draw a circle through P, Q, R: then its radius

$$=\frac{\overline{RQ}}{2 \sin QPR} \simeq \frac{\text{arc } RPQ}{2 \sin QPS} \simeq \frac{2\delta s}{2 \sin QPS}.$$

Therefore when $\delta s \to 0$, the radius of the limiting circle

$$= \rho = \text{Lt}\, \frac{\delta s}{\sin QPS} = \text{Lt}\, \frac{\delta s}{\delta \psi} = \frac{ds}{d\psi}.$$

This limiting circle is called the *circle of curvature* at P; it touches the curve at P and its curvature is equal to that of the curve. The radius of this circle, which equals $\frac{ds}{d\psi}$, is called the *radius of curvature* at P and equals the reciprocal of the curvature.

For a circle, radius a (see Fig. 169), we have $s = a\psi$ and $\frac{ds}{d\psi} = a$. Also the curvature

$$= \frac{d\psi}{ds} = \frac{1}{a}.$$

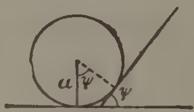

Fig. 169.

The circle is the only plane curve of constant curvature.

Note. The circle of curvature at any point P of a curve is the circle which touches the curve at P and which bends round at the same rate as the curve is bending at P.

(i) The radius of curvature at the point (x, y) on $y = f(x)$ is given by

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

We have $$\frac{dy}{dx} = \tan\psi;$$

$$\therefore \frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx} = \sec^2\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx} = \sec^3\psi \frac{d\psi}{ds} \text{ since } \frac{ds}{dx} = \sec\psi;$$

$$\therefore \frac{d^2y}{dx^2} = (\sec^2\psi)^{\frac{3}{2}} \frac{d\psi}{ds} = (1 + \tan^2\psi)^{\frac{3}{2}} \frac{d\psi}{ds}.$$

$$\rho = \frac{ds}{d\psi} = \frac{(1 + \tan^2\psi)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Note. If $\frac{dy}{dx}$ is small, e.g. along a rod bending slightly under its own weight,

$$\rho \triangleq \frac{1}{\frac{d^2y}{dx^2}}.$$

(ii) The radius of curvature at a point on a curve given by its pedal equation

$$r=f(p) \text{ is } \rho = r\frac{dr}{dp}.$$

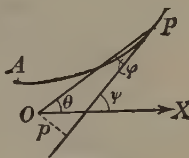

Fig. 170.

With the usual notation

$$\frac{1}{\rho}=\frac{d\psi}{ds}=\frac{d}{ds}(\theta+\phi)=\frac{d\theta}{ds}+\frac{d\phi}{ds}.$$

$$\text{Now } \frac{p}{r}=\sin\phi, \quad \therefore \frac{1}{r}\frac{dp}{dr}-\frac{p}{r^2}=\cos\phi\frac{d\phi}{dr};$$

$$\therefore \frac{1}{r}\frac{dp}{dr}=\frac{r\sin\phi}{r^2}+\frac{dr}{ds}\cdot\frac{d\phi}{dr}=\frac{1}{r}\cdot\frac{rd\theta}{ds}+\frac{d\phi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}=\frac{1}{\rho};$$

$$\therefore \rho=r\frac{dr}{dp}.$$

Note. If we wish to find the radius of curvature for a curve given in polar coordinates, it is best to start by finding the pedal equation and then use $\rho=r\frac{dr}{dp}$.

Example 6.

Find the radius of curvature at the point (r, θ) on the parabola

$$\frac{2a}{r}=1+\cos\theta.$$

We have $\quad \frac{a}{r}=\cos^2\frac{\theta}{2} \quad \text{or} \quad \log a-\log r=2\log\cos\frac{\theta}{2};$

$$\therefore -\frac{1}{r}\frac{dr}{d\theta}=-\tan\frac{\theta}{2};$$

$$\therefore \cot\phi=\tan\frac{\theta}{2}=\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \text{ or } \phi=\frac{\pi}{2}-\frac{\theta}{2};$$

$$\therefore p=r\sin\phi=r\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=r\cos\frac{\theta}{2};$$

$$\therefore p^2=r^2\cos^2\frac{\theta}{2}=r^2\times\frac{a}{r}=ar;$$

$$\therefore 2p\frac{dp}{dr}=a;$$

$$\therefore \rho=r\frac{dr}{dp}=r\times\frac{2p}{a}=\frac{2r}{a}\times\sqrt{ar}=2\sqrt{\left(\frac{r^3}{a}\right)}.$$

EXAMPLES XV e

1. Find the radius of curvature at (1, 1) on $y^2=x^3$.

2. Find the radius of curvature at the origin on

$$\text{(i) } y=ax^2, \quad \text{(ii) } y=ax^2+bx^3.$$

3. Find the radius of curvature at the point (x, y) on

(i) the hyperbola $xy=c^2$,

(ii) the catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$.

4. Find the radius of curvature at the point (r, θ) on

(i) the equiangular spiral $r=a e^{\theta \cot \alpha}$.

(ii) the cardioid $r=a(1+\cos\theta)$.

5. If in the cardioid $r=a(1+\cos\theta)$, s is measured from $\theta=0$ and if $0<\theta<\pi$, prove that $s^2+9\rho^2=16a^2$.

6. Find the radius of curvature at the point θ on the cycloid given by $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

7. Find the radius of curvature at the point t on the curve

$$x=a\cos^3 t, \quad y=a\sin^3 t.$$

8. If the curve $y=f(x)$ touches the x-axis at the origin, prove that the radius of curvature at the origin $=\underset{x\to 0}{\text{Lt}}\ \frac{x^2}{2y}$.

9. If $\frac{d^2y}{dx^2}$ vanishes and changes sign in passing through the point P on the curve $y=f(x)$, what geometrical interpretation is suggested by the formula for ρ?

10. If OY is the perpendicular from the origin O to the tangent at any point P on the curve, prove with the usual notation that $PY=\frac{dp}{d\psi}$.

$$\left[\text{Note } \rho=\frac{ds}{d\psi}=r\frac{dr}{dp}.\right]$$

11. Prove that the radius of curvature at the point (x, y) on the parabola $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ is $\frac{2}{ab}(ax+by)^{\frac{3}{2}}$.

12. Find the radius of curvature at the point $(a\cos\phi, b\sin\phi)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

VI. Surface of a Solid of Revolution

When the arc AB rotates about OX and traces out the surface of a solid of revolution, the chord PQ of the small element δs, where arc $AP = s$, will trace out the surface of the frustum of a cone whose area is given by (circumference of mean section) × (slant height).

$\therefore$ the chord PQ traces out a surface of area δS where

$$\delta S = 2\pi\left(\frac{y + (y + \delta y)}{2}\right)\overline{PQ},$$

S being the area traced out by the arc AP.

$$\therefore \frac{\delta S}{\overline{PQ}} = \pi\,[2y + \delta y].$$

$$\therefore \frac{\delta S}{\widehat{PQ}} \times \frac{\widehat{PQ}}{\overline{PQ}} = \pi\,[2y + \delta y],$$

$$\therefore \frac{\delta S}{\delta s} \doteq 2\pi y,$$

and in the limit when $\delta s \to 0$

$$\frac{dS}{ds} = 2\pi y, \quad \therefore\ S = \underline{\int 2\pi y\, ds.}$$

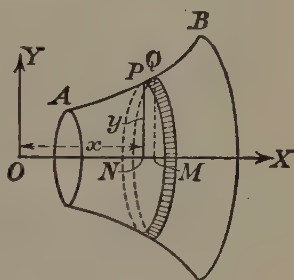

Fig. 171.

It is important to notice that this formula is *not* $\int 2\pi y\,dx$: the area is *not* traced out by an element δx but by δs.

If the curve is given in polar coordinates and if the axis of rotation is the initial line, this formula may be written

$$S = 2\pi \int r \sin\theta\, ds.$$

Example 7.

Find the area of a spherical cap of height h. Let $AB = h$, and let R be the radius of the sphere; then if the coordinates of P are (x, y), we have $x^2 + y^2 = R^2$.

$$\therefore 2x\,dx + 2y\,dy = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left|\frac{dy}{dx}\right|^2} = \sqrt{1 + \frac{x^2}{y^2}} = \frac{R}{y}.$$

$$\therefore y\,ds = R\,dx.$$

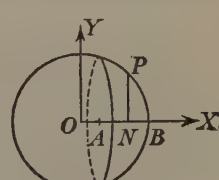

Fig. 172.

$$\therefore\ S=\int 2\pi y\,ds=\int_{R-h}^{R} 2\pi R\,dx$$

$$=2\pi R\int_{R-h}^{R} dx=2\pi R\Big[x\Big]_{R-h}^{R}$$

$$=\underline{2\pi R h.}$$

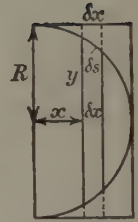

Fig. 173.

In particular, the area of the surface of the whole sphere

$$=2\pi R\,(2R)=4\pi R^2.$$

Note that the relation $y\,ds=R\,dx$ shows that

$$2\pi y\,ds=2\pi R\,dx,$$

i.e. the area of the belt of surface of sphere = area of corresponding belt of the circumscribing cylinder. [Archimedes' Theorem.]

Theorems of Pappus and Guldin

(1) If a plane area A revolves about an axis lying in its plane but not intersecting it, the volume of the solid so generated is measured by

$A \times$ length of path of centroid of area.

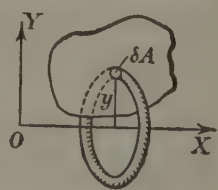

Fig. 174.

Consider a small element δA at a distance y from the axis OX.

Volume of ring traced out by $\delta A \doteq 2\pi y\,\delta A$.

$\therefore$ volume of whole solid $\doteq 2\pi\Sigma y\,\delta A$.

But if $\bar{y}$ is the distance from OX of the centroid of A, we have

$$\bar{y}\doteq\frac{\Sigma\,(y\,\delta A)}{\Sigma\,\delta A}\doteq\frac{\Sigma\,(y\,\delta A)}{A}.$$

$$\therefore\ \Sigma\,(y\,\delta A)\doteq\bar{y}\times A.$$

$\therefore$ when $\delta A\to 0$ we have

$$\text{volume of whole solid}=2\pi\int y\,dA$$

$$\text{and } \int y\,dA=\bar{y}\times A.$$

$\therefore$ volume of whole solid $=2\pi\bar{y}\times A$

$= A\times$ length of path of centroid.

Example 8.

Find the volume of an anchor ring formed by the rotation of a circle of radius a about a line at distance d from the centre, where $d > a$.

The centroid traces out a circle of radius d.

$\therefore$ the volume of the anchor ring $= \pi a^2 \times 2\pi d$
$= 2\pi^2 a^2 d.$

A simple example of an anchor ring or *tore*, as it is also called, is the ring on a curtain-pole.

(2) When an arc CD of a plane curve of length S revolves about an axis lying in its plane but not intersecting it, the area of the surface so generated is measured by

$S \times$ length of path of centroid of arc.

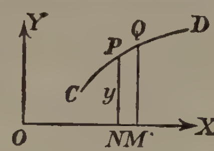

Fig. 175.

Consider an element $PQ = \delta s$ of CD which is rotated about the axis OX and let the ordinate $PN = y$.

Then it is shown on p. 249 that the area of the surface traced out by $PQ \triangleq 2\pi y \delta s$ and the area traced out by $CD = \int 2\pi y\, ds = 2\pi \int y\, ds$.

Also if the centroid of arc CD is at a distance $\bar{y}$ from OX, we have

$$\bar{y} \triangleq \frac{\Sigma (y\,\delta s)}{\Sigma\,\delta s} \text{ and } \bar{y} = \frac{\int y\,ds}{S} \text{ or } \bar{y} \times S = \int y\,ds.$$

$\therefore$ the area traced out by $CD = 2\pi\bar{y} \times S$
$= S \times$ length of path of centroid of arc.

Example 9.

Find the area of the surface of an anchor ring formed by the rotation of a circle of radius a about a line at distance d from the centre, where $d > a$.

The centroid traces out a circle of radius d.

$\therefore$ area of surface of anchor ring $= 2\pi a \times 2\pi d$
$= 4\pi^2 ad.$

Note. (i) The statements in (1) and (2) also hold good for a rotation through any angle less than four right angles.

(ii) If the expressions for the volume or the area of a surface of revolution are known, these theorems enable us to find the distance of the centroid from the axis of rotation of the area or curve from which they have been generated.

EXAMPLES XV f

1. Find the curved surface of a right cone, base-radius r inches, slant side l inches.

2. Find the surface generated by the parabola $y^2=4ax$ when the portion from $(0, 0)$ to (x, y) revolves about OX.

3. Find the surface of a reflector whose shape is a paraboloid of depth 10 inches and greatest diameter 2 feet 6 inches.

4. Show that, if ϕ varies, the point $x=a\sin^3\phi$, $y=a\cos^3\phi$ traces out the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. Hence find the area of the surface generated by rotating this curve about the x-axis.

5. For the cardioid $r=a(1+\cos\theta)$, prove that $\frac{ds}{dr}=-\operatorname{cosec}\frac{\theta}{2}$; hence find the area of the surface formed by rotating the curve about the initial line.

6. For the "figure-of-eight" curve $r^2=a^2\cos 2\theta$, prove that

$$\frac{ds}{dr}=-\operatorname{cosec} 2\theta;$$

hence find the area of the surface formed by rotating the curve about the initial line.

7. If the portion of the catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$ between $x=0$ and $x=a$ is rotated (i) about the x-axis, (ii) about the y-axis, find the area of the surface generated.

8. Interpret geometrically the expression $\int 2\pi(y\cos a-x\sin a)\,ds$, where a is a constant and the integral is taken along a curve of which δs is an element.

9. A curtain ring has an external diameter of 3 inches and its cross section is a circle of diameter $\frac{1}{2}$ inch. Find the area of its surface.

10. The triangle ABC right-angled at B is rotated about BC to generate a circular cone; $BC=h$, $AB=r$; deduce from Pappus' theorem the volume and area of the surface of the cone.

11. (i) By assuming the surface of a sphere $=4\pi r^2$, find by Pappus' theorem the position of the centre of gravity of a semicircular arc.

(ii) A semicircle of radius $\frac{1}{2}$ inch is placed with its diameter parallel to OX and 1 inch away from it and its rim on the side of the diameter remote from OX; find the area of the surface generated by rotating it about OX.

12. (i) Deduce from Archimedes' expression for the area of a spherical cap (p. 250) the position of the centre of gravity of an arc of a circle radius r subtending an angle $2a$ at the centre.

(ii) An arc of a circle of radius r is of length $2ra$; find the area of the surface generated by rotating the arc about its chord.

13. By assuming the volume of a sphere $=\frac{4}{3}\pi r^3$, deduce the position of the centroid of a semicircular lamina.

14. A hemispherical bowl of radius 2 feet contains water; find the depth of the water when half the surface is wetted.

15. AB is the diameter of a semicircle, centre O; C is the mid-point of arc AB; OC is produced to D so that $OC = CD = 2$ inches. The semicircular arc is rotated about the line through D parallel to AB; find the area of the surface so obtained.

VII. Two Important Curves

(i) *The Catenary.*

A catenary is the curve assumed by a uniform flexible chain when suspended from two points.

With the notation of Fig. 176, we see that the portion CP of the chain is in equilibrium under the action of the tension T at P, the horizontal tension T_0 at C and the weight ws, where w is the weight of unit length.

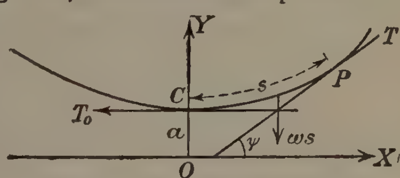

Fig. 176.

Resolving horizontally and vertically,

$$T\cos\psi = T_0 \text{ and } T\sin\psi = ws. \quad \therefore \tan\psi = \frac{ws}{T_0}.$$

Put $T_0 = wa$, i.e. suppose the tension at C to be the weight of a length a of chain: then $\frac{s}{a} = \tan\psi$ or $s = a\tan\psi$.

We wish to deduce the Cartesian equation from this result.

We have $\frac{ds}{dy} = a\sec^2\psi\frac{d\psi}{dy}$ but $\frac{ds}{dy} = \operatorname{cosec}\psi$. $\therefore \frac{dy}{d\psi} = a\frac{\sin\psi}{\cos^2\psi}$.

$$\therefore\ y = a\int \frac{\sin\psi}{\cos^2\psi}\,d\psi = a\sec\psi,$$

if we choose $y=a$ when $\psi=0$. Similarly

$$\frac{ds}{dx} = a\sec^2\psi\,\frac{d\psi}{dx}\ \text{ but }\ \frac{ds}{dx} = \sec\psi.$$

$$\therefore\ \frac{dx}{d\psi} = a\sec\psi,$$

and
$$x = a\int\frac{d\psi}{\cos\psi} = a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right),$$

see p. 227, if $x=0$ when $\psi=0$.

$$\therefore\ e^{\frac{x}{a}} = \tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\ \text{ and }\ e^{-\frac{x}{a}} = \cot\left(\frac{\pi}{4}+\frac{\psi}{2}\right).$$

$$\therefore\ e^{\frac{x}{a}} + e^{-\frac{x}{a}} = \frac{\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)} + \frac{\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)} = \frac{1}{\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}$$

$$= \frac{2}{\sin\left(\frac{\pi}{2}+\psi\right)} = 2\sec\psi = 2\frac{y}{a}.$$

$$\therefore\ y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

Note: the equation of the catenary may be expressed in any of the forms obtained above

(i) $s = a\tan\psi$.

(ii) $y = a\sec\psi$.

(iii) $x = a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$.

(iv) $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

From (i) and (ii) we deduce the result

$$s^2 = a^2\tan^2\psi = a^2(\sec^2\psi - 1) = y^2 - a^2.$$

(ii) *The Cycloid.*

When a circle rolls along a straight line, the path traced out by any point P on its circumference is called a *cycloid*.

Let C be the centre and a the radius of the circle which is rolling along the line OX and let the origin O be the starting-point of P so that arc $MP = MO$.

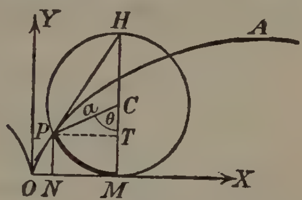

Fig. 177.

When the circle has turned through an angle θ (i e. $\angle MCP=\theta$), let the coordinates of P be x, y so that $ON=x$, $NP=y$.

Then

$$x = ON = OM - NM = \widehat{PM} - PT = a\theta - a\sin\theta = a(\theta - \sin\theta),$$

and $\quad y = NP = MT = MC - TC = a - a\cos\theta = a(1-\cos\theta).$

The equation of the cycloid is therefore $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$.

Since the circle is rolling without slipping, M is instantaneously at rest and $\therefore$ the circle is instantaneously turning about M.

$\therefore$ P is moving at right angles to PM.

But if MH is a diameter, $\angle MPH$ is a right angle.

$\therefore$ PH is the tangent at P.

This result should be verified by Calculus methods.

The other chief property of the cycloid is as follows:

If A is the vertex of the curve (i.e. the point at which the tangent is parallel to OX), the arc $AP =$ twice the chord HP.

We have

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta = a^2(2-2\cos\theta) = 4a^2\sin^2\frac{\theta}{2}.$$

$$\therefore \frac{ds}{d\theta} = 2a\sin\frac{\theta}{2} \text{ and arc } PA = \int_\theta^\pi 2a\sin\frac{\theta}{2}\,d\theta = 4a\cos\frac{\theta}{2}.$$

But $$\angle PHM = \tfrac{1}{2}\angle PCM = \frac{\theta}{2}.$$

$$\therefore \text{arc } PA = 4a\cos PHM = 2HM\cos PHM = 2PH.$$

EXAMPLES XV g

The Catenary

1. Show that the tension at any point of the catenary equals the weight of a piece of chain whose length equals the ordinate at that point. If a chain hangs over two smooth pegs, what is the equilibrium position?

2. In Fig. 176, prove that arc $CP = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

3. Show that the area bounded by the x-axis, the ordinates $x=0$ and $x=x_1$ and the catenary is $a\sqrt{(y_1{}^2 - a^2)}$.

4. If in Fig. 176, N is the foot of the perpendicular from P to OX and if Y is the foot of the perpendicular from N to the tangent at P, prove that $NY = a$ and that arc $CP = YP$.

5. If the normal at P meets the x-axis at G, prove that $PG = \frac{y^2}{a}$.

The Cycloid

6. In Fig. 177 prove that the tangent at P makes an angle $\frac{\pi}{2} - \frac{\theta}{2}$ with OX.

7. Prove that the length of the arc of one arch of the cycloid $= 8a$.

8. Prove that the area enclosed by OX and one arch $= 3\pi a^2$.

9. If s is measured from A and if the initial line is the tangent at A, prove that the equation of the cycloid can be written in the form

$$s = 4a \sin \psi.$$

10. Prove that the radius of curvature at P equals $2PM$.

11. A particle is sliding down a smooth inverted cycloid (i.e. Fig. 177 upside down); if s is measured from A, the equation of motion is

$$\frac{d^2s}{dt^2} = -g \sin \psi; \quad \text{where } g \text{ is a constant;}$$

prove that this can be written

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

If it starts from rest at $s = 4a$, find its velocity in passing through A. Also if the solution is of the form $s = p\cos(nt + \epsilon)$, determine p, n, ϵ and hence find the time taken to reach A.

12. With the data of Ex. 11, if the mass of the particle is m lbs., the pressure on the curve R lbs. at any point P is given by

$$\frac{R-m\cos\psi}{m}=\frac{v^2}{g\rho},$$

where ρ is the radius of curvature at P and v is the speed with which it passes P; express R in terms of ψ.

REVISION PAPERS 18—24

R. 18

1. Differentiate (i) $\cos^2(3x+2)$, (ii) $\sin^{-1}\left(\frac{2x+3}{4}\right)$.

2. For the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, prove that the tangent to the curve makes with OX an angle ϕ such that $\tan\phi=\cot\frac{\theta}{2}$.

3. Use differentials to find $\frac{dy}{dx}$ when (i) $x^2+y^2=a^2$, (ii) $\sqrt{x}+\sqrt{y}=\sqrt{a}$, (iii) $xy=C$.

4. For an alternating current the flow of electricity C is given by $CR+L\frac{dC}{dt}=E\cos pt$, where E, p, R, L are constants; prove that the equation is satisfied by a solution of the form $C=a\cos pt+b\sin pt$ and find a, b.

5. A cup of coffee at temperature 100° C. is placed in a room whose temperature is 15° and it cools 60° in 5 minutes. Find the temperature after another 5 minutes.

R. 19

1. Differentiate (i) $\tan^{-1}\left(\frac{a}{x}\right)$, (ii) $\operatorname{cosec}^{\frac{1}{2}}(1-x)$.

Evaluate $\int_0^a (a^2-x^2)^{\frac{5}{2}}\,dx$.

2. Find the equation of the tangent at the point ϕ to the curve given by $x=a\cos^3\phi$, $y=a\sin^3\phi$. Find also the radius of curvature at this point.

3. A segment of a sphere stands on a base of radius a and its height is h; show that a section of the segment by a plane parallel to the base and at distance y from it is of area $\pi\left[a^2-\frac{y}{h}(a^2-h^2)-y^2\right]$ and deduce that the volume of the segment is $\frac{\pi h}{6}(3a^2+h^2)$.

4. A triangular wall bracket ABC is fastened to a wall with A vertically above B, and C below the level of B; the rods AC, BC stretch so that C descends vertically a small distance δx; express δx in terms of δa, δb; a, b, c being the lengths of BC, CA, AB.

5. Find the position of the centre of gravity of the solid formed by the revolution about OX of $y=5\sin 2x$ between $x=0$ and $x=\frac{\pi}{4}$.

R. 20

1. Evaluate (i) $\frac{d}{dx}\log(1-x)$; (ii) $\frac{d}{dx}\log(\cot 2x)$;

(iii) $\int_{-2}^{-1}\frac{dx}{1-2x}$; (iv) $\int_0^\infty e^{-x}\,dx$.

2. Find a differential equation which expresses the fact that the rate at which a liquid passes through a conical paper filter is proportional to the depth of liquid left, the original volume being V_0.

3. The velocity of a body is given by $v=80e^{-\frac{t}{10}}$ in ft./sec. units; find the distance travelled in 5 seconds.

4. Use Simpson's rule to evaluate $\int_5^6 \frac{dx}{x}$ and compare your result with the corresponding logarithm.

5. A rectangular trough 5 feet deep, 3 feet broad is closed by a heavy uniform sluice gate hinged at its upper edge and resting on the bottom of the trough at an angle 60° to the horizontal. If the trough is filled with water (1 cu. ft. weighs 62 lbs.), find the minimum weight of the gate if it is to remain closed.

R. 21

1. Write down the differentials of (i) $x^{-\frac{1}{2}}$; (ii) e^{ax}; (iii) $\log(a-bx)$; (iv) $\tan x$.

2. If $\theta=Ae^{-\lambda t}\sin(at+b)$, where A, a, b are constants, prove that

$$\frac{d^2\theta}{dt^2}+2\lambda\frac{d\theta}{dt}+(\lambda^2+a^2)\,\theta=0.$$

3. Calculate the area between the curves $y=xe^{-x}$, $y=xe^{x}$ and the line $x=1$.

4. Show that the expression $\tan^{-1}x-\frac{x}{1+x^2}$ is always positive as x changes from 0 to $\frac{\pi}{2}$.

5. A body moves in a straight line so that when its speed is v ft./sec. its retardation is $-kv$ ft./sec.^{2} Initially its speed is v_0 ft./sec.; find how far it travels in t seconds. If $v_0=20$, $k=\frac{1}{4}$, find how long it takes to travel 50 feet, and show that it never reaches a point 80 feet from the starting-point.

R. 22

1. Evaluate (i) $\int (2x^2-5)^2\,dx$, (ii) $\displaystyle\int \frac{6x-5}{3x^2-5x+7}\,dx$,

(iii) $\int (\cos 3\theta+\cos 5\theta)\,d\theta$, (iv) $\int \cos 3\theta \cos 5\theta\,d\theta$.

2. Calculate the amount of £100 in 5 years at 4 per cent. per annum at *continuous* compound interest.

3. A beam supported at its two ends and loaded at the middle sags a distance x given by $x=\dfrac{kl^3}{bt^3}$, where b, l, t are the breadth, length, thickness respectively of the beam and k is a constant. If the possible errors in b, l, t are each 0·1 per cent., find the approximate maximum percentage error in the deflection x.

4. A body is placed 3 feet from a point O and moves so that its speed is $2x$ ft./sec. when it is x feet from O. Find how far it goes in 2 secs., and the time it takes to go 50 feet.

5. If $P(x_1, y_1)$ is a point on $y=ke^{\frac{x}{a}}$, PN its ordinate, PT the tangent and PG the normal meeting OX at T, G respectively, prove that NT is constant and that $NG=\dfrac{y_1{}^2}{a}$.

R. 23

1. Evaluate (i) $\displaystyle\int_0^\pi \sin^2 2\theta\,d\theta$, (ii) $\displaystyle\int_0^1 \frac{x^2}{4-x^2}\,dx$.

2. If $e^{(p-q)t}=\dfrac{\lambda\,(p-x)}{\mu\,(q-x)}$ express $\dfrac{dx}{dt}$ in terms of x.

3. The motion of a galvanometer needle is given by $\theta=4e^{-\frac{t}{2}}\sin 3t$, where θ is the angle the needle makes with the zero position after t seconds. Find the angular velocity of the needle at any time t and prove that the angles corresponding to the extreme positions of the oscillating needle form a G.P. and find its common ratio.

4. For the curve $ay^2=x^3$ if the length of arc measured from the origin to the point (x, y) is s, prove that

$$\left(\frac{27s}{8a}+1\right)^2=\left(1+\frac{9x}{4a}\right)^3.$$

5. A rope is wound twice round a post and held by a force of 20 lbs. at one end. If the coefficient of friction equals 0·4, find what force is required to make the rope slip.

R. 24

1. If $\frac{dx}{dt}=k(a-x)(b-x)$ and if $x=0$ when $t=0$, express t in terms of x.

2. A pane of glass 1 cm. thick absorbs 5 % of the light incident on it. What percentage of light will pass through a sheet of similar glass 1·6 cms. thick?

3. Evaluate (i) $\int \frac{dx}{(x^2-4)(x+3)}$, (ii) $\int_a^b \frac{dx}{\sqrt{\{(x-a)(b-x)\}}}$.

[In (ii) use the substitution $x=a\cos^2\theta+b\sin^2\theta$.]

4. Find the area between the hyperbola $r^2\cos 2\theta=a^2$ and the lines

$$\theta=\pm\frac{\pi}{6}.$$

5. A variable current i flows along a conductor according to the law $i=a\sin(\omega t)$, where t measures the time. Find the mean value of i^2 for the period $t=0$ to $t=\frac{2\pi}{\omega}$.

MISCELLANEOUS EXAMPLES 24—31

M. 24

1. The formula $y=R\frac{x}{l-x}$ gives the electrical resistance (y) of a coil as measured on a form of Wheatstone Bridge, l being the fixed length of a German silver wire and x the adjustable resistance of a key from one end of the wire. Suppose that with a given value of R an error a is made in x, show that the error in y is approximately $\frac{yla}{x(l-x)}$ and hence show that the percentage error in y for a given value of a will be least when the key is in the middle of the wire.

2. Find the condition to be satisfied by the coefficients to secure that

$$x^3+ax^2+bx+c$$

shall increase as x increases for all values of x.

3. The area bounded by the curve $y=f(x)$, the axes of x and y and the ordinate $x=a$ being supposed to rotate about the axis OY, prove that the volume generated is given by the integral $\int_0^a 2\pi\, xy\, dx$.

The segment of the parabola $y^2=4ax$ bounded by the latus rectum $x=a$ being rotated about the tangent at the vertex, find the volume generated and show as well as you can by sketches the shape of the solid generated. (Army.)

4. In an experiment it was found that the pressure of air on a moving plane was proportional to $\sin\theta\left(1+\dfrac{1}{1+\tan^2\theta}\right)$. Prove that this is equal to $1{\cdot}25\sin\theta+0{\cdot}25\sin 3\theta$.

Construct the curve $r=1{\cdot}25\sin\theta+0{\cdot}25\sin 3\theta$ from $\theta=0°$ to $\theta=180°$. Find the maximum and minimum values of r between these limits and the corresponding values of θ. (Army.)

5. Find the area of the curve $r=1{\cdot}25\sin\theta+0{\cdot}25\sin 3\theta$.

M. 25

1. Sketch the curves $y=e^x$ and $y=xe^x$ between $x=-1$ and $x=1$. Calculate the area enclosed between portions of these two curves and OY.

2. Find the volume of the solid formed by a complete revolution about OX of that portion of $y=a\sin\dfrac{\pi x}{b}$ between $x=0$ and $x=b$.

3. Find (i) $\dfrac{d}{dx}\left(x^{\frac{1}{x}}\right)$, (ii) $\dfrac{d^n}{dx^n}\log_e(x^2+4x+3)$.

4. Find the maxima and minima values of $3\cos^2 x+2(1+\sin x)^2$, distinguishing maxima from minima.

5. For the hyperbola $x^2-y^2=a^2$ prove that the tangent and the radius vector from the centre at any point make complementary angles with OX.

Find the polar equation of this curve.

M. 26

1. A cylinder 25 cms. high and 100 sq. cms. in cross-section has a hole at the bottom. The rate at which water pours out of the hole is proportional to the square root of the depth of water in the cylinder and is 1 cc. per sec. when the cylinder is full. How long will the cylinder take to empty itself?

2. The cross-section of a trough is a parabola with vertex downwards, the latus rectum 4 feet in length lying in the surface. Find the pressure on the end of the trough when it is full of water.

3. The floor of a level quarry is 60 feet below the top. A load of 5 cwt. is drawn slowly across the bottom of the quarry by a wire from the top at A. If the coefficient of friction is 0·2 and x is the distance of the load from the face of the quarry vertically below A, prove that the work done in dragging it from $x=120$ to $x=90$ is $-\int_{120}^{90} \frac{x}{12+x}\,dx$ and that this equals 27 ft. cwts.

4. A tumbler is full of wine. A man sips $\frac{1}{n}$th of the liquid and water is then added until the tumbler is full. If he does this n times where n is infinite, what fraction of the mixture is then wine?

5. The base of a solid is the area between the lines $y=2x$ and $y=-2x$ for the values of x from 0 to 5, and the height at any point whose coordinates are (x, y) is $1+\frac{x}{10}(x-5)$ Find its volume, if the unit on each axis is 1 inch.

M. 27

1. Prove that the volume of a segment of a sphere less than a hemisphere, the radius of whose base is a and whose height is h, is approximately $\frac{1}{2}\pi a^2 h$ and that if $h=\frac{a}{4}$ the error is approximately $2\,\%$.

2. Find the greatest and least values of $\frac{8}{5+3\sin 2\theta}$.

3. A certain function of x is equal to ax^2 for values of x less than 1 and to $-ax^2+bx-1$ for values of x greater than 1. Find the values of the constants a and b in order that there may be no discontinuity or abrupt changes of slope in the graph of the function at the point $x=1$. With these values of a and b find the values of x for which the function is zero. (Cambridge University.)

4. In a battle between two opposing forces the number of casualties per unit time in either force is proportional to the number of men in the opposing force, the constant factor representing the efficiency of the opposing force which is different for the two sides.

If m and n denote the numbers in the two forces at any time t, a and b the factors representing their efficiency, write down the values of $\frac{dm}{dt}$ and $\frac{dn}{dt}$, and prove that $am^2-bn^2=\text{const.}$

If the two forces contain 10,000 and 5000 men respectively and the smaller force is twice as efficient as the larger, which side will win and how many men will survive if the battle is continued to the bitter end? (Army.)

5. The air pressure on an aeroplane is approximately proportional to $A \sin \theta + B \sin 3\theta$. Find the condition that this expression should have a maximum value for some value of θ between 0° and 90° if $A > 0$.

Find A and B if the expression is a maximum when $\theta = 45°$ and the expression equals 1 when $\theta = 90°$. Find also the maximum value. (Army.)

M. 28

1. Find the rate at which the Napierian logarithm of a number increases compared with the rate at which the number increases when the number is 3. Compare with the tables.

2. The equation of a curve is $s^2 = 4ay$, where s is its length measured from the origin. Find the area of the surface of the solid obtained by the revolution of an arc of length s_1 measured from O of this curve about OX.

3. Show that in the curve $r = a(1 - \cos \theta)$ the angle between the radius vector and the tangent is $\dfrac{\theta}{2}$.

4. A stone is thrown with a velocity of 80 ft./sec. at an angle of 60° to the horizontal. Taking its highest point as origin and assuming that the path is a parabola, find the equation of the path and the whole length of the path.

5. A quantity of steam expands so as to satisfy the law $pv^{1 \cdot 13} = \text{const.}$ Find the work done in expanding from $v = 3$ to $v = 10$ cu. ft. Given $p = 8640$ lbs./sq. ft. when $v = 1$.

M. 29

1. Show that the curve $s = \sqrt{8ay}$ may also be written in the form $s = 4a \sin \psi$, where ψ is the angle the tangent makes with OX.

2. Find the area of $r = a\theta$ from $\theta = 0$ to $\theta = 2\pi$.

3. A triangular partition in a trough is in the shape of an equilateral triangle of side 2 ft. with its base horizontal. If the level of the water on one side is 1 ft. and on the other side 8 ins. above the vertex, find the magnitude and point of action of the resultant thrust on the partition due to the water.

4. Find the area between the curve $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$, the axis OX and the lines $x=0$, $x=a$.

5. Two large parallel plate condensers at a distance a apart are charged with equal quantities of electricity of opposite sign. The potential V at a point on the positive plate is

$$\int_0^\infty 2\pi\sigma\left(1-\frac{r}{\sqrt{(r^2+a^2)}}\right)dr.$$

Prove that this equals $2\pi a\sigma$, where σ is a constant

M. 30

1. To measure the distance of an observer O from an accessible point A, a rod 10 ft. long is set up at A at right angles to OA with its mid-point at A and the angle θ subtended by the rod at O is then measured. Find OA in terms of θ and find the error in OA caused by an error $\delta\theta$ in θ. Evaluate when $\theta=72'$ and $\delta\theta=1'$. (Army.)

2. A rhombus consisting of four equal uniform freely-jointed rods, each of weight W, is suspended from an angular point which is connected with the opposite angular point by an elastic string. Show by Virtual Work that in the position of equilibrium the tension in the string equals $2W$.

3. Fig. 178 shows the plan and elevation of a roof. $ABCD$ is part of the curved surface of a cylinder whose radius is a and whose axis is level with the ridge AB and vertically over CD. The other three portions of the roof are also parts of cylinders of the same radius a. Calling the angle AOP, θ, the angle AOR, $\theta+\delta\theta$, and the length CD, b, express the length of PQ and the approximate area of the strip $PQSR$ in terms of a, b, θ and $\delta\theta$. Find by integration the total area of the portion $ABCD$ of the roof. (Army.)

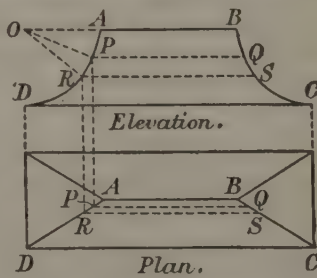

Fig. 178.

4. A water main 4 ft. in diameter is closed by a circular gate. Find the water pressure on it when the main is half full.

5. Find the area of the surface generated by the revolution of a semi-circular arc about a tangent at its mid-point.

M. 31

1. ABC is a triangle of area s; if a and b remain constant but the other parts vary, prove that

$$\frac{\delta s}{s} = -\frac{\delta A \cos C}{\sin A \cdot \cos B}.$$

2. $ABCD$ is a square. Along AB take any point H. Along BC, CD, DA measure BK, CL, DM respectively each equal to AH. Join AK, BL, CM, DH intersecting one another and forming an inner square $PQRS$. Prove that the ratio of the mean area of $PQRS$ to the area $ABCD$ is

$$\log \frac{e}{2}.$$

3. Find the area between $y = ae^{-bx} \sin x$ and OX from $x=0$ to π.

If $a=1$, $b=0{\cdot}1$, prove that the decrease of area when b is increased by a small quantity λ is approximately $2{\cdot}6\lambda$. (Cambridge University.)

4. Show for the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ that $\frac{ds}{dx} = \sqrt[3]{\left(\frac{a}{x}\right)}$, and if s is measured from $(0, a)$ find s in terms of x.

5. The magnetic intensity at a point distant r from a long linear conductor carrying a current C is $\int_{-\infty}^{+\infty} \frac{Cr\,dx}{(r^2+x^2)^{\frac{3}{2}}}$; prove this equals $\frac{2C}{r}$.

CHAPTER XVI

COMPLEX NUMBER AND THE HYPERBOLIC FUNCTIONS

The treatment of certain standard integrals is much simplified by using functions which have a certain analogy with the trigonometric functions: this analogy can best be seen by introducing complex numbers; and because a knowledge of the use of complex numbers is required in all higher branches of mathematics and physics, it is desirable to discuss very shortly their meaning.

Complex Number

If $x^2 = -1$, there is no real value of x which satisfies the equation, because there is no real number whose square is -1. In order therefore to say that every quadratic has a root we introduce what we call *imaginary numbers* and we write

$$x = +\sqrt{-1} \text{ or } -\sqrt{-1}.$$

Our definition of the symbol $\sqrt{-1}$ is the statement

$$\sqrt{-1} \times \sqrt{-1} = -1.$$

By using this symbol, we can now say that every quadratic has two roots.

Thus take the equation $x^2 + a^2 = 0$, we have

$$x^2 = -a^2 = \sqrt{-1} \times \sqrt{-1}\, a^2,$$

$$\therefore\ x = +\sqrt{-1}\, a \text{ or } x = -\sqrt{-1}\, a;$$

or take for example $x^2 + 6x + 13 = 0$, we have

$$x^2 + 6x + 9 = -4 \text{ or } (x+3)^2 = (2\sqrt{-1})^2,$$

$$\therefore\ x + 3 = \pm 2\sqrt{-1} \text{ or } x = -3 \pm 2\sqrt{-1}.$$

Such a number as $-3 + 2\sqrt{-1}$ is called a *complex number* and the number $2\sqrt{-1}$ is called a *pure imaginary*. When we say

that $x = -3 + 2\sqrt{-1}$ is a root of $x^2 + 6x + 13 = 0$, we simply mean that if we substitute this expression for x and use the ordinary rules of algebra to simplify and whenever $\sqrt{-1} \times \sqrt{-1}$ occurs we write -1 in its place, then $x = -3 + 2\sqrt{-1}$ will make the expression $x^2 + 6x + 13$ equal to zero.

Thus if $x = -3 + 2\sqrt{-1}$, $x^2 = (-3 + 2\sqrt{-1})(-3 + 2\sqrt{-1})$,

$$\therefore x^2 = 9 - 6\sqrt{-1} - 6\sqrt{-1} + 4\sqrt{-1} \times \sqrt{-1}$$
$$= 9 - 12\sqrt{-1} - 4 = 5 - 12\sqrt{-1},$$
$$\therefore x^2 + 6x + 13 = 5 - 12\sqrt{-1} + 6(-3 + 2\sqrt{-1}) + 13$$
$$= 5 - 12\sqrt{-1} - 18 + 12\sqrt{-1} + 13 = 0.$$

Complex numbers are therefore introduced to secure a continuity or generality of statement which would otherwise be unattainable. They constitute a new kind of number, or more accurately a complex number is really a pair of numbers associated together in a particular way. We shall *assume* that the same algebraic operations which can be performed on real numbers can also be performed on complex numbers and shall use this assumption to attach meanings to functions of complex numbers. This is precisely the method adopted in the introduction of fractional indices: we there assumed the general law $x^m \times x^n = x^{m+n}$ and deduced that since $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x$ the symbol $x^{\frac{1}{2}}$ means $\sqrt{x}$.

We shall now proceed to attach a meaning to the symbol $e^{\theta\sqrt{-1}}$ on the *assumption* that it obeys the same formal laws of algebra as e^x does when x is real.

In what follows, for the sake of brevity, we shall denote $\sqrt{-1}$ by i.

To show that the same meaning* must be attached to the symbol $e^{i\theta}$ as to the expression $\cos\theta + i\sin\theta$

or $\cos\theta + i\sin\theta = e^{i\theta}$.

* The symbol $e^{i\theta}$ really represents a many-valued function, but here it is used to represent the principal value of the function which equals 1 when $\theta = 0$.

Let $$y = \cos\theta + i\sin\theta,$$

$$\therefore \frac{dy}{d\theta} = -\sin\theta + i\cos\theta = i^2\sin\theta + i\cos\theta \quad \text{since } i^2 = -1$$

$$= i(i\sin\theta + \cos\theta) = iy,$$

$$\therefore \frac{d\theta}{dy} = \frac{1}{iy} \text{ or } i\frac{d\theta}{dy} = \frac{1}{y},$$

$$\therefore i\theta = \int\frac{dy}{y} = \log y + c.$$

But when $\theta = 0$, $y = \cos 0 + i\sin 0 = 1$;

$$\therefore 0 = \log 1 + c = 0 + c.$$

$$\therefore i\theta = \log y,$$

or $$y = e^{i\theta}.$$

$$\therefore \underline{\cos\theta + i\sin\theta = e^{i\theta}}.$$

Put $\theta = -\phi$ and we have

$$\cos(-\phi) + i\sin(-\phi) = e^{-i\phi},$$

$$\therefore \cos\phi - i\sin\phi = e^{-i\phi}.$$

But the letter used does not make any difference;

$$\therefore \cos\theta - i\sin\theta = e^{-i\theta}.$$

By addition and subtraction we have

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \text{ and } \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

and by division $$\tan\theta = \frac{1}{i}\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}.$$

These relations lead up to the introduction of certain new functions called the *hyperbolic functions*.

Corresponding to $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ we write $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$.

Corresponding to $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ we write $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$.

Corresponding to $\tan\theta = \frac{1}{i}\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$ we write $\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$.

For euphony "sinh" is pronounced "shin" and "tanh" is pronounced "than."

In addition we write

$$\frac{1}{\sinh\theta}=\text{cosech}\,\theta,\quad \frac{1}{\cosh\theta}=\text{sech}\,\theta,\quad \frac{1}{\tanh\theta}=\coth\theta.$$

"cosech" and "sech" are pronounced "coshec" and "shec" respectively.

The hyperbolic functions have properties very similar to the trigonometric functions.

Example 1.

Prove that (i) $\cosh^2\theta-\sinh^2\theta=1$;

(ii) $\sinh 2\theta=2\sinh\theta\cosh\theta$.

$$\text{(i)}\quad \cosh^2\theta-\sinh^2\theta=\tfrac{1}{4}(e^\theta+e^{-\theta})^2-\tfrac{1}{4}(e^\theta-e^{-\theta})^2$$
$$=\tfrac{1}{4}(e^{2\theta}+2e^0+e^{-2\theta})-\tfrac{1}{4}(e^{2\theta}-2e^0+e^{-2\theta})$$
$$=\tfrac{1}{4}\times 4e^0=1.$$

$$\text{(ii)}\quad 2\sinh\theta\cosh\theta=2\times\tfrac{1}{2}(e^\theta-e^{-\theta})\times\tfrac{1}{2}(e^\theta+e^{-\theta})$$
$$=\tfrac{1}{2}(e^{2\theta}-e^{-2\theta})=\sinh 2\theta.$$

The relation $\cosh^2\theta-\sinh^2\theta=1$ shows that the point $(a\cosh\theta, b\sinh\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. From this fact the functions derive their names.

The following simple rule (the justification for which is suggested in Example 14, p. 271) enables any hyperbolic function formula to be deduced from the corresponding trigonometrical formula.

In any trigonometrical formula, replace $\sin\theta$, $\cos\theta$, $\tan\theta$ **by** $\sinh\theta$, $\cosh\theta$, $\tanh\theta$ **and wherever a product of sines or an** ***implied*** **product of sines occurs, change the sign of the term.***

Thus

$$\cos 2\theta=\cos^2\theta-\sin^2\theta \quad\text{becomes}\quad \cosh 2\theta=\cosh^2\theta+\sinh^2\theta$$

and

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}\ \text{becomes}\ \tanh(\theta+\phi)=\frac{\tanh\theta+\tanh\phi}{1+\tanh\theta\tanh\phi}$$

for
$$\tan\theta\tan\phi\equiv\frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}$$

is an *implied* product of sines.

All such formulae can be proved by simple substitution from the definitions.

* This rule is due to Mr. G. Osborn (*Math. Gazette*, July, 1920).

EXAMPLES XVI a

1. Simplify:

(i) $(x+iy)(x-iy)$; (ii) $(1+i)^2$; (iii) $\left(\frac{1+i\sqrt{3}}{2}\right)^2$; (iv) $\left(\frac{1-i\sqrt{3}}{2}\right)^3$.

2. Solve the equations:

$$\text{(i) } x^2+4=0; \quad \text{(ii) } x^2+8x+25=0.$$

3. Show that 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$ are the three roots of $x^3=1$.

4. If a, b are real numbers such that $(a-3)+i(b-5)=0$, find their values.

5. Find the real values of r and θ if $r(\cos\theta+i\sin\theta)=3+4i$.

6. Find the roots of $x^3-x^2+2=0$, given that $x=1+i$ is one root.

7. Prove that

(i) $(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$;

(ii) $(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)=\cos(\theta+\phi)+i\sin(\theta+\phi)$;

and use (ii) to show that if n is any positive integer

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

8. Use Ex. 7 to show that if q is a positive integer $\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}$ is one of the qth roots of $\cos\theta+i\sin\theta$: show that $\cos\frac{\theta+2\pi}{q}+i\sin\frac{\theta+2\pi}{q}$ is another qth root of $\cos\theta+i\sin\theta$. How many are there?

9. Use Ex. 8 to show that $\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}$ is one of the values of

$$(\cos\theta+i\sin\theta)^{\frac{p}{q}},$$

where p, q are positive integers.

10. Prove that

$$(\cos\theta+i\sin\theta)^{-1}\equiv\frac{1}{\cos\theta+i\sin\theta}=\cos\theta-i\sin\theta=\cos(-\theta)+i\sin(-\theta)$$

and deduce that if n is any positive number

$$(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta.$$

11. Use the relation $\quad \cos\theta+i\sin\theta=e^{i\theta}$

to prove that $\quad \cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n$.

12. Use the result of Ex. 11 to prove that

(i) $\cos 3\theta=4\cos^3\theta-3\cos\theta$; (ii) $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

13. Simplify:

$$\text{(i)}\ \frac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta - i\sin 3\theta}; \quad \text{(ii)}\ \frac{1+\cos\theta + i\sin\theta}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}.$$

14. Use the exponential forms for $\cos\theta$ and $\sin\theta$ to prove that

$$\text{(i)}\ \cos(i\theta) = \cosh\theta; \quad \text{(ii)}\ \sin(i\theta) = i\sinh\theta;$$

and deduce the mnemonic given above for obtaining formulae for the hyperbolic functions.

15. Prove that

(i) $\sinh(-x) = -\sinh x$; (ii) $\cosh(-x) = \cosh x$;
(iii) $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$; (iv) $\cosh x \geqslant 1$.

16. What is the value when $x = 0$ of

(i) $\sinh x$, (ii) $\cosh x$, (iii) $\tanh x$?

17. If $y = \sinh^{-1} x$, prove that

(i) $2x = e^y - e^{-y}$; (ii) $e^{2y} - 2x \, . \, e^y - 1 = 0$; (iii) $e^y = x + \sqrt{1+x^2}$.

Hence show that $\sinh^{-1} x = \log(x + \sqrt{1+x^2})$.

18. Use the method of Ex. 17 to prove that

$$\text{(i)}\ \cosh^{-1} x = \pm\log(x + \sqrt{x^2-1}); \quad \text{(ii)}\ \tanh^{-1} x = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

19. Prove that

$$\text{(i)}\ \frac{\cosh 2x - 1}{\cosh 2x + 1} = \tanh^2 x; \quad \text{(ii)}\ \frac{1+\tanh x}{1-\tanh x} = e^{2x};$$

$$\text{(iii)}\ \sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x.$$

20. Prove that

$$(\cosh\theta + \sinh\theta)^n = \cosh(n\theta) + \sinh(n\theta)$$

and simplify $(\cosh\theta - \sinh\theta)^n$.

21. Prove that

$$\text{(i)}\ \operatorname{sech}^2\theta = 1 - \tanh^2\theta; \quad \text{(ii)}\ \operatorname{cosech}^2\theta = \coth^2\theta - 1;$$

$$\text{(iii)}\ \coth\theta + \operatorname{cosech}\theta = \coth\frac{\theta}{2}.$$

22. If $\sinh x = \tan\theta$, express $\cosh x$ and $\tanh x$ in terms of θ.

23. Prove that

(i) $\sec\theta = \cosh[\log(\tan\theta + \sec\theta)]$;
(ii) $\cot\theta = \sinh[\log(\cot\theta + \operatorname{cosec}\theta)]$.

24. Find approximately the value of x for which $\cosh x = 2$.

Differentiation and Integration of $\sinh\theta$ **and** $\cosh\theta$

To prove

$$\frac{d}{d\theta}(\sinh\theta) = \cosh\theta \quad\text{and}\quad \frac{d}{d\theta}(\cosh\theta) = \sinh\theta$$

we have $$\frac{d}{d\theta}(\sinh\theta) = \frac{d}{d\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right) = \frac{e^{\theta}+e^{-\theta}}{2} = \cosh\theta$$

and $$\frac{d}{d\theta}(\cosh\theta) = \frac{d}{d\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) = \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta.$$

Reversing these results, we have

$$\int\cosh\theta\,.\,d\theta = \sinh\theta \quad\text{and}\quad \int\sinh\theta\,.\,d\theta = \cosh\theta.$$

Integrals of functions involving $\sqrt{a^2+x^2}$ can often be simplified by the substitution $x=a\sinh\theta$ and those involving $\sqrt{x^2-a^2}$ by putting $x=a\cosh\theta$.

Example 2.

Integrate $$\int\frac{dx}{\sqrt{a^2+x^2}}.$$

Put $x=a\sinh\theta$, $\therefore\ dx=a\cosh\theta\,.\,d\theta$

and $$a^2+x^2=a^2+a^2\sinh^2\theta=a^2(1+\sinh^2\theta)=a^2\cosh^2\theta.$$

$$\therefore\ \text{the integral} = \int\frac{a\cosh\theta\,.\,d\theta}{a\cosh\theta} = \int d\theta=\theta.$$

But $$\sinh\theta=\frac{x}{a}.$$

$\therefore$ using the inverse notation

$$\theta=\sinh^{-1}\left(\frac{x}{a}\right).$$

$$\therefore\ \int\frac{dx}{\sqrt{a^2+x^2}}=\sinh^{-1}\left(\frac{x}{a}\right).$$ [See p. 271, no. 17, and p. 228.]

Example 3.

Find $$\int\sqrt{x^2-a^2}\,dx.$$

Put $x=a\cosh\theta$, $\therefore\ dx=a\sinh\theta\,.\,d\theta$

and $$x^2-a^2=a^2(\cosh^2\theta-1)=a^2\sinh^2\theta.$$

$$\therefore \text{ the integral} = \int a \sinh\theta \,.\, a \sinh\theta \,.\, d\theta$$
$$= a^2 \int \sinh^2\theta \,.\, d\theta$$
$$= \frac{a^2}{2}\int (\cosh 2\theta - 1)\, d\theta$$
$$= \frac{a^2}{2}\left(\tfrac{1}{2}\sinh 2\theta - \theta\right)$$
$$= \frac{a^2}{2}(\sinh\theta\cosh\theta - \theta)$$
$$= \frac{a^2}{2}\left[\frac{\sqrt{x^2-a^2}}{a} \,.\, \frac{x}{a} - \cosh^{-1}\left(\frac{x}{a}\right)\right]$$
$$= \tfrac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right).$$

EXAMPLES XVI b

1. Prove that

(i) $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$; (ii) $\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x.$

2. Prove that $\dfrac{d}{dx}\left[\log\left(\tanh\dfrac{x}{2}\right)\right] = \operatorname{cosech} x.$

3. If $y = \sinh^{-1} x$ or $x = \sinh y$, prove that

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$$

and deduce $$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}}.$$

What is $$\frac{d}{dx}\left[\sinh^{-1}\frac{x}{a}\right]?$$

4. Use the method of Ex. 3 to prove that

(i) $\dfrac{d}{dx}[\cosh^{-1} x] = \pm\dfrac{1}{\sqrt{x^2-1}}$; (ii) $\dfrac{d}{dx}[\tanh^{-1} x] = \dfrac{1}{1-x^2}.$

What is $$\frac{d}{dx}\left[\tanh^{-1}\frac{x}{a}\right]?$$

5. If $$y = a\cosh nx + b\sinh nx,$$

prove that $$\frac{d^2y}{dx^2} - n^2y = 0.$$

6. What are the values of

$$\int \sinh(nx)\,dx \quad \text{and} \quad \int \cosh(nx)\,dx?$$

Prove that $$\int \tanh(nx)\,dx = \frac{1}{n}\log[\cosh(nx)].$$

7. Prove that $$\int \cosh^2\theta\,.\,d\theta = \frac{\theta}{2} + \tfrac{1}{4}\sinh 2\theta.$$

8. What are the values of

$$\int \operatorname{sech}^2 nx\,.\,dx \quad \text{and} \quad \int \operatorname{cosech}^2 nx\,.\,dx?$$

9. Fig. 179 represents part of the graph of $x^2 - y^2 = 1$. If $ON = \cosh\theta$, prove that $NP = \sinh\theta$ and that the area

$$ANP = \int_0^\theta \sinh^2\theta\,d\theta.$$

Write down the area of triangle ONP and deduce that the area of the sector

$$OAP = \tfrac{1}{2}\theta.$$

What is the corresponding result for the sector of the circle

$$x^2 + y^2 = 1?$$

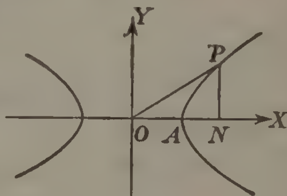

Fig. 179.

10. Prove that

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right).$$

11. Evaluate: (i) $\displaystyle\int \frac{dx}{\sqrt{(9x^2+1)}}$; (ii) $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$.

12. Evaluate:

$$\text{(i)} \int \frac{dx}{\sqrt{[(x+a)^2 - b^2]}}; \quad \text{(ii)} \int \frac{dx}{\sqrt{(x^2+10x+16)}}.$$

13. Evaluate $\displaystyle\int \frac{x^2\,dx}{\sqrt{(1+x^2)}}$ by putting $x = \sinh\theta$.

14. Express the product $\sinh(ax)\sinh(bx)$ as a difference and hence evaluate

$$\int \sinh(ax)\sinh(bx)\,dx.$$

15. If $\cosh u = \sec\theta$, prove that $\dfrac{du}{d\theta} = \sec\theta$ and that

$$u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

16. The equation $y=c\cosh\dfrac{x}{c}$ represents a catenary (see p. 254). If

$$\left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2,$$

express s in terms of x, given that $s=0$ when $x=0$.

17. For the catenary $y=c\cosh\dfrac{x}{c}$, prove with the usual notation:

$$\text{(i) } y=c\sec\psi;\quad \text{(ii) } y^2=c^2+s^2;\quad \text{(iii) } x=c\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right).$$

18. Find the radius of curvature at the point (x, y) on the catenary

$$y=c\cosh\frac{x}{c}.$$

19. Evaluate:

$$\text{(i) } \int x\sinh x\,dx;\quad \text{(ii) } \int e^x\cosh x\,dx.$$

20. Evaluate: (i) $\displaystyle\int\frac{dx}{\sinh x}$; (ii) $\displaystyle\int\sinh^{-1}x\,dx$.

21. Prove that $\dfrac{d}{dx}\left\{\tan^{-1}\left(\tanh\dfrac{x}{2}\right)\right\}=\tfrac{1}{2}\operatorname{sech} x$.

22. Evaluate:

$$\text{(i) } \int_1^2\frac{dx}{\sqrt{(x^2+1)}};\quad \text{(ii) } \int_2^3\frac{dx}{\sqrt{(x^2-1)}};\quad \text{(iii) } \int_5^9\sqrt{(x^2+144)}\,dx.$$

23. Evaluate:

$$\text{(i) } \int_0^1\cosh^2 x\,dx;\quad \text{(ii) } \int_0^1 x\cosh x\,dx.$$

24. A body weighing 20 lbs. falls from rest under gravity in a medium in which the resistance is $\dfrac{v^2}{20}$ lbs. wt., where v ft. per sec. is the speed after t seconds. Prove that

(i) $\dfrac{dv}{dt}=\dfrac{g}{400}(400-v^2)$; (ii) if the final (limiting) speed is v_1 ft. per sec., $t=\dfrac{v_1}{g}\tanh^{-1}\left(\dfrac{v}{v_1}\right)$. What is v_1?

Find also the time taken to acquire a velocity of 10 ft. per sec. and the distance the body has then fallen.

CHAPTER XVII

EXPANSIONS IN SERIES

It is often a matter of practical utility to be able to replace a complicated function of x by one of a simpler type (such as part of a power series in x) which for a special range of values of x approximates in value to the original function.

Example 1.

If x is positive and small compared with 1, $\sqrt[3]{(1+x)} \doteq 1+\frac{1}{3}x$ with error less than $\frac{1}{9}x^2$.

We have $$(1+\tfrac{1}{3}x)^3 = 1+x+\tfrac{1}{3}x^2+\tfrac{1}{27}x^3 > 1+x$$

and $$(1+\tfrac{1}{3}x-\tfrac{1}{9}x^2)^3 = 1+x-\frac{x^3(15-x^2)}{81}-\frac{x^6}{729}$$
$$< 1+x \qquad \text{since } x>0.$$

$\therefore$ $\sqrt[3]{(1+x)}$ lies between $1+\frac{1}{3}x$ and $1+\frac{1}{3}x-\frac{1}{9}x^2$.

$\therefore$ $\sqrt[3]{(1+x)} \doteq 1+\frac{1}{3}x$ with error less than $\frac{1}{9}x^2$.

This example shows the degree of accuracy attained when the comparatively complicated function $\sqrt[3]{(1+x)}$ is replaced by the first two terms of a power series. By using the Binomial theorem, we can obtain as many terms as we wish in the expansion of $\sqrt[3]{(1+x)}$ as a power series.

If we know that any given function of x **can be expressed** as a power series in x, we can obtain the form of the series by a calculus method.

Example 2.

Given that $$e^x \equiv a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots,$$
where the coefficients a_0, a_1, ... are independent of x, show how to find them.

Since the identity is true for all values of x, we may put $x=0$, then
$$e^0 = a_0, \quad \therefore\ a_0 = 1.$$

Differentiate with respect to x,

$$\therefore\ e^x \equiv a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots;$$

in this identity, put $x=0$,

$$\therefore\ e^0 = a_1 \text{ or } a_1 = 1.$$

Differentiate again,

$$\therefore\ e^x \equiv 2a_2 + 2\,.\,3a_3x + 3\,.\,4a_4x^2 + \dots;$$

put $x=0$, $\qquad \therefore\ e^0 = 2a_2$ or $a_2 = \frac{1}{2}$;

repeat the process,

$$\therefore\ e^x \equiv 2\,.\,3a_3 + 2\,.\,3\,.\,4a_4x + \dots;$$

put $x=0$, $\qquad \therefore\ e^0 = 2\,.\,3a_3$ or $a_3 = \dfrac{1}{2\,.\,3}$,

similarly $a_4 = \dfrac{1}{2\,.\,3\,.\,4}$, $a_5 = \dfrac{1}{2\,.\,3\,.\,4\,.\,5}$, and so on.

$$\therefore\ a_0 = 1,\ a_1 = 1,\ a_2 = \frac{1}{\underline{|2}},\ a_3 = \frac{1}{\underline{|3}},\ a_4 = \frac{1}{\underline{|4}},\ \dots,\ a_n = \frac{1}{\underline{|n}},\ \dots$$

and $$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots + \frac{x^n}{\underline{|n}} + \dots.$$

Note. We have made two serious assumptions in establishing the expansion of e^x as a power series of x:

(i) that it is possible to express e^x as an infinite series in x;

(ii) that the differentiation of this power series can be effected term by term.

Both of these assumptions are true in the case of the function e^x, but there are functions for which neither assumption is true (e.g. $\log x$ cannot be expanded as an infinite power series in x) and therefore the proof as given above is incomplete.

We shall now apply the method of this example to any function $f(x)$. The following notation will be used:

$$f'(x),\ f''(x),\ f'''(x),\ \dots,\ f^n(x),\ \dots$$

will represent

$$\frac{d}{dx}f(x),\ \frac{d^2}{dx^2}f(x),\ \frac{d^3}{dx^3}f(x),\ \dots,\ \frac{d^n}{dx^n}f(x),\ \dots,$$

and $$f'(a),\ f''(a),\ \ldots,\ f^n(a),\ \ldots$$
represent the result of substituting a for x, *after* the differentiation has been completed, in
$$f'(x),\ f''(x),\ \ldots,\ f^n(x),\ \ldots,$$
and in particular $f^n(0)$ represents the result of substituting 0 for x in $f^n(x)$ *after* differentiation.

Maclaurin's Theorem

If $f(x)$ is a function which possesses successive differential coefficients and if it can be expanded as an infinite series in powers of x, then the expansion is
$$f(x) \equiv f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \ldots + \frac{x^n}{\underline{|n}} f^n(0) + \ldots.$$

Let $$f(x) \equiv a_0 + a_1\frac{x}{\underline{|1}} + a_2\frac{x^2}{\underline{|2}} + a_3\frac{x^3}{\underline{|3}} + \ldots + a_n\frac{x^n}{\underline{|n}} + \ldots.$$

[Numerical factors are inserted in the denominators to make the work symmetrical, this does not make the form less general.]

In this identity, put $x = 0$, $\therefore\ f(0) = a_0$.

Differentiate with respect to x,
$$\therefore\ f'(x) = a_1 + a_2\frac{x}{\underline{|1}} + a_3\frac{x^2}{\underline{|2}} + \ldots + a_n\frac{x^{n-1}}{\underline{|n-1}} + \ldots,$$
put $x = 0$, $$\therefore\ f'(0) = a_1.$$

Differentiate again,
$$\therefore\ f''(x) = a_2 + a_3\frac{x}{\underline{|1}} + \ldots + a_n\frac{x^{n-2}}{\underline{|n-2}} + \ldots,$$
put $x = 0$, $$\therefore\ f''(0) = a_2;$$
clearly we can repeat this process as often as we wish and we obtain
$$a_3 = f'''(0),\ a_4 = f''''(0),\ \ldots,\ a_n = f^n(0),\ \ldots,$$
$$\therefore\ f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \ldots + \frac{x^n}{\underline{|n}} f^n(0) + \ldots.$$

Note. We have made the same assumptions in this proof as are noted on p. 277.

Example 3.

By assuming Maclaurin's Theorem, prove that

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \dots.$$

Here

$$\begin{array}{lll} f(x) = \ \ \sin x, & \therefore f(0) = & \sin 0 = 0; \\ f'(x) = \ \ \cos x, & \therefore f'(0) = & \cos 0 = 1; \\ f''(x) = -\sin x, & \therefore f''(0) = & -\sin 0 = 0; \\ f'''(x) = -\cos x, & \therefore f'''(0) = & -\cos 0 = -1; \\ f''''(x) = \ \ \sin x, & \therefore f''''(0) = & \sin 0 = 0; \end{array}$$

and this cycle of values now recurs indefinitely,

$$\therefore \sin x = 0 + x + 0 - \frac{x^3}{\lfloor 3} + 0 + \frac{x^5}{\lfloor 5} \dots$$

$$= x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots.$$

EXAMPLES XVII a

By assuming Maclaurin's Theorem, obtain the expansions in Examples 1—9:

1. $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \dots.$

2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \dots + x^n$, if n is a positive integer.

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ (valid only if $-1 < x \leqslant 1$).

4. $\sinh x = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots.$

5. $\cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots.$

6. $a^x = 1 + x \log a + \frac{x^2 (\log a)^2}{\lfloor 2} + \frac{x^3 (\log a)^3}{\lfloor 3} + \dots$

7. $\sec x = 1 + \frac{x^3}{2} + \frac{5x^4}{24} + \dots.$

8. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots.$

9. $\tan^{-1} x = x - \frac{x^3}{3} + \dots$ (valid only if $-1 \leqslant x \leqslant 1$).

10. Expand $e^x \cos x$ in powers of x up to x^5.

11. Find the first three terms in the expansion of $\log(1+e^x)$.

12. Expand $e^{\tan x}$ up to x^3.

13. Expand $\log(1+\sin x)$ up to x^4.

14. Show that $\log\left(\frac{\sin x}{x}\right) = -\frac{x^2}{6} - \frac{x^4}{180} \dots$.

15. Write down the expansion for e^{ix}; then equate real and imaginary parts in the relation $\cos x + i \sin x = e^{ix}$.

16. [Taylor's Theorem.] Assuming that $f(x+h)$ can be expanded as a power series in h, prove by the method used above for Maclaurin's Theorem that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \dots + \frac{h^n}{\underline{|n}} f^n(x) + \dots.$$

Other methods of expansion or summation

When a function of x is expressed as an infinite series in powers of x, it is true *under certain conditions* to say that the differential coefficient of the function equals the result of differentiating the series term by term and the integral of the function is equal to the result of integrating the series term by term. It is beyond the scope of this book to investigate what these conditions are, we shall however illustrate the process by two examples.

Example 4.

Obtain a series for $\sin^{-1}(x)$ if $-1<x<1$.

By the binomial theorem, since $-1<x<1$,

$$\frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1\,.\,2}x^4 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\,.\,2\,.\,3}x^6 + \dots$$

$$= 1 + \tfrac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \dots.$$

Integrate with respect to x,

$$\therefore \int \frac{dx}{\sqrt{(1-x)^2}} = \sin^{-1} x = c + x + \tfrac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7} + \dots.$$

If when $x=0$ we take the principal value of $\sin^{-1} x$ as 0, we have $c=0$.

$$\therefore \sin^{-1} x = x + \tfrac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7} + \dots.$$

Example 5.

If $-1<x<1$, sum the series

$$1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}+\ldots.$$

Since $-1<x<1$, we have

$$\frac{1}{(1-x)^2}=(1-x)^{-2}=1+2x+3x^2+\ldots+nx^{n-1}+\ldots$$

$$\therefore\ \frac{x}{(1-x)^2}=x+2x^2+3x^3+\ldots+nx^n+\ldots;$$

differentiate with respect to x,

$$\therefore\ \frac{1}{(1-x)^2}+\frac{2x}{(1-x)^3}=1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}+\ldots,$$

$$\therefore\ \text{required series}=\frac{1-x+2x}{(1-x)^3}=\frac{1+x}{(1-x)^3}.$$

EXAMPLES XVII b

1. Obtain a series for $\tan^{-1}x$ by using

$$\tan^{-1}x=\int_0^x\frac{dx}{1+x^2}$$

when $-1<x<1$.

2. Obtain a series for $\log(1+x)$ if $-1<x<1$ from

$$\log(1+x)=\int_0^x\frac{dx}{1+x}.$$

3. What relation can be obtained by integrating $\dfrac{2}{1-x^2}$?

4. If $-1<x<1$, sum the series

$$1.2+2.3x+3.4x^2+4.5x^3+\ldots.$$

5. If $-1<x<1$, sum the series

$$\frac{x^2}{1.2}-\frac{x^3}{2.3}+\frac{x^4}{3.4}-\ldots.$$

6. If $-1<x<1$, sum the series

$$1+\frac{1.3}{2}x^2+\frac{1.3.5}{2.4}x^4+\frac{1.3.5.7}{2.4.6}x^6+\ldots.$$

7. Sum the series

$$1-3\frac{x^2}{\underline{|2}}+5\frac{x^4}{\underline{|4}}-7\frac{x^6}{\underline{|6}}+\ldots.$$

8. Sum the series

$$1+\frac{2x^2}{\underline{|3}}+\frac{3x^4}{\underline{|5}}+\frac{4x^6}{\underline{|7}}+\frac{5x^8}{\underline{|9}}+\dots.$$

9. From the series

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots,$$

where $-1<x<1$, deduce the sum of the series

$$x-\frac{2x^3}{3}+\frac{3x^5}{5}-\frac{4x^7}{7}+\dots.$$

10. If $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+x=e^{-2t}$ and if when $t=0$, $x=0$ and $\dfrac{dx}{dt}=0$, show that

$$x=\tfrac{1}{2}t^2-\tfrac{2}{3}t^3+\tfrac{11}{24}t^4-\dots.$$

11. Expand $\sin^2 x$ in powers of x. [Use multiple angles.]

12. Expand $\sin x\cos x$ in powers of x.

13. [Huygen's rule.] If a is the chord of the arc AB of a circle and if b is the chord of half the arc AB, then the length of the arc is approximately $\frac{1}{3}(8b-a)$. If the arc AB subtends θ radians at the centre of the circle, show that the error per cent. is approximately $\dfrac{\theta^4}{75}$.

14. Expand $\tanh^{-1}x$ as a power series in x if $-1<x<1$.

15. Expand $\log[x+\sqrt{(1+x^2)}]$ as a power series in x if $-1<x<1$.

In the work of this chapter we have so far given no proof of the *existence* of the various expansions considered. We have merely shown what forms they must take, *if they exist at all.*

Now when we say that a function $f(x)$ is equal to an infinite series in powers of x,

$$a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots,$$

what we mean is that the difference between the value of $f(x)$ and the value of the n terms

$$a_0+a_1x+\dots+a_{n-1}x^{n-1}$$

can be made as small as we please by taking a sufficiently large value of n for any value of x in the range of values of x considered.

If therefore we can obtain an expression for this difference, we may then be able to see if it satisfies this essential test: if it does not do so, the expansion is untrue.

Example.

Under what conditions does

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots\to\infty\,?$$

Now by actual division or by taking the sum of a G.P.

$$1+x+x^2+\ldots+x^{n-1}=\frac{1-x^n}{1-x},$$

$\therefore$ the difference $\dfrac{1}{1-x}-\{1+x+x^2+\ldots+x^{n-1}\}=\dfrac{1}{1-x}-\dfrac{1-x^n}{1-x}=\dfrac{x^n}{1-x}.$

The test therefore is this: can we find a value of n such that $\dfrac{x^n}{1-x}$ is less than any given amount however small, for this and all greater values of n? If we can do so, we may regard

$$1+x+x^2+\ldots+x^{n-1}$$

as an approximation for $\dfrac{1}{1-x}$ and the approximation can be made as close as we please by taking n large enough.

Now obviously we cannot always do this; for example if $x=2$,

$$\frac{x^n}{1-x}=\frac{2^n}{1-2}=-2^n,$$

and so the larger n becomes the greater is the error in taking

$$1+x+x^2+\ldots+x^{n-1}$$

to represent $\dfrac{1}{1-x}$.

But if x is any fraction *between* 1 and -1, for example $\frac{9}{10}$, we can make

$$\frac{x^n}{1-x}=10\,(0{\cdot}9)^n$$

as small as we please by taking n large enough.

We therefore say that if x is any fraction between 1 and -1, the function $\dfrac{1}{1-x}$ and the infinite series $1+x+x^2+\ldots$ are equal; they are not equal if $x\geqq 1$ or if $x\leqq -1$.

Before trying to apply this test to Maclaurin's Theorem, we must establish the following result.

If when x varies from a to $a+h$ where $h>0$, the greatest and least values of the continuous function $\phi(x)$ are M and m, then

$$Mh > \int_a^{a+h} \phi(x)\,dx > mh.$$

Let A, B be the points on the graph of $y=\phi(x)$ whose abscissae are a, $a+h$, and let GN, gn represent the greatest and least values of $\phi(x)$ between A and B.

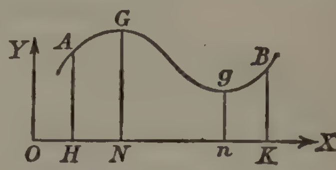

Fig. 180.

Then

$$HK = OK - OH = a+h-a = h,$$

$$GN = M, \quad gn = m.$$

$$\int_a^{a+h} \phi(x)\,dx = \text{the curvilinear area } HAGgBK.$$

This is obviously less than $GN \times HK$ or Mh and obviously greater than $gn \times HK$ or mh,

$$\therefore\ Mh > \int_a^{a+h} \phi(x)\,dx > mh.$$

We shall now use the method of integrating by parts to obtain the expansion of a function.

Example 6.

Expand e^a in powers of a, where $a>0$.

We have $e^a - 1 = \int_0^a e^x\,dx = -\int_{x=0}^a e^x\,d(a-x)$

$$= -\Big[e^x(a-x)\Big]_0^a + \int_0^a (a-x)\,d(e^x)$$

$$= -(0-a) + \int_0^a (a-x)\,e^x\,dx$$

$$= a - \int_0^a e^x\,d\left(\frac{(a-x)^2}{2}\right)$$

$$= a - \left[e^x \cdot \frac{(a-x)^2}{2}\right]_0^a + \int_0^a \frac{(a-x)^2}{2}\,d(e^x)$$

$$= a - \left(0 - \frac{a^2}{2}\right) + \int_0^a \frac{(a-x)^2}{2}\,e^x\,dx$$

$$= a + \frac{a^2}{2} - \int_0^a e^x\,d\left(\frac{(a-x)^3}{2 \cdot 3}\right)$$

$$=a+\frac{a^2}{2}-\left[e^x\cdot\frac{(a-x)^3}{2\cdot 3}\right]_0^a+\int_0^a\frac{(a-x)^3}{2\cdot 3}\,d(e^x)$$

$$=a+\frac{a^2}{2}+\frac{a^3}{2\cdot 3}+\int_0^a\frac{(a-x)^3}{2\cdot 3}\,e^x dx.$$

Clearly we can repeat this process as often as we like, we thus obtain

$$e^a=1+a+\frac{a^2}{\lfloor 2}+\frac{a^3}{\lfloor 3}+\frac{a^4}{\lfloor 4}+\ldots+\frac{a^n}{\lfloor n}+\int_0^a\frac{(a-x)^n}{\lfloor n}\,e^x dx.$$

We cannot evaluate the integral obtained here but we can find values between which it must lie: for when x varies from 0 to a, $(a-x)^n$ lies between a^n and 0, also e^x lies between e^0 and e^a or 1 and e^a.

$\therefore\ \dfrac{(a-x)^n}{\lfloor n}\,e^x$ lies between 0 and $\dfrac{a^n}{\lfloor n}\,e^a$.

$\therefore\ \displaystyle\int_0^a\frac{(a-x)^n}{\lfloor n}\,e^x dx>0$ and $<\dfrac{a^n e^a}{\lfloor n}\times a$ or $\dfrac{a^{n+1}e^a}{\lfloor n}$.

$$\therefore\ e^a>1+a+\frac{a^2}{\lfloor 2}+\ldots+\frac{a^n}{\lfloor n}$$

and
$$e^a<1+a+\frac{a^2}{\lfloor 2}+\ldots+\frac{a^n}{\lfloor n}+\frac{a^{n+1}e^a}{\lfloor n}.$$

$\therefore$ the error in replacing e^a by $1+a+\dfrac{a^2}{\lfloor 2}+\ldots+\dfrac{a^n}{\lfloor n}$ is less than $\dfrac{a^{n+1}e^a}{\lfloor n}$.

We need $\therefore$ only consider whether this error can be made as small as we please by taking a large enough value for n.

Now if we take $(n+2)$ terms instead of $(n+1)$ terms the error is $\dfrac{a^{n+2}e^a}{\lfloor n+1}$, i.e. it is $\dfrac{a}{n+1}\times$ the former error; and so as soon as $n>2a$, the addition of each successive term makes the new error less than half the old error and $\therefore$ we can make the error as small as we please by carrying out the halving process a sufficient number of times.

We therefore say that for *all* positive values of a

$$e^a=1+a+\frac{a^2}{\lfloor 2}+\frac{a^3}{\lfloor 3}+\ldots+\frac{a^n}{\lfloor n}+\ldots\to\infty.$$

A similar proof may be used if $a<0$; in that case the integral may be negative, but the error remains less than the positive value of $\dfrac{a^{n+1}e^a}{\lfloor n}$.

We shall use the method of integrating by parts followed in this example to give a more accurate statement of Maclaurin's Theorem.

Maclaurin's Theorem

If as x varies from 0 to a, $f(x)$ is continuous and possesses continuous successive differential coefficients, then

$$f(a) = f(0) + af'(0) + \frac{a^2}{\underline{|2}} f''(0) + \dots + \frac{a^n}{\underline{|n}} f^n(0) + \int_0^a \frac{(a-x)^n}{\underline{|n}} f^{n+1}(x)\,dx.$$

We have

$$\begin{aligned}
f(a) - f(0) &= \int_0^a f'(x)\,dx = -\int_{x=0}^{a} f'(x)\,d(a-x) \\
&= -\Big[f'(x)(a-x)\Big]_0^a + \int_0^a (a-x)\,d\,[f'(x)] \\
&= -[0 - af'(0)] + \int_0^a (a-x) f''(x)\,dx \\
&= af'(0) - \int_0^a f''(x)\,d\left(\frac{(a-x)^2}{2}\right) \\
&= af'(0) - \left[f''(x) \,.\, \frac{(a-x)^2}{2}\right]_0^a + \int_0^a \frac{(a-x)^2}{2}\,d\,[f''(x)] \\
&= af'(0) - \left[0 - \frac{a^2}{2} f''(0)\right] + \int_0^a \frac{(a-x)^2}{2} f'''(x)\,dx \\
&= af'(0) + \frac{a^2}{2} f''(0) - \int_0^a f'''(x)\,d\left[\frac{(a-x)^3}{2\,.\,3}\right] \\
&= af'(0) + \frac{a^2}{2} f''(0) - \left[f'''(x) \,.\, \frac{(a-x)^3}{2\,.\,3}\right]_0^a + \int_0^a \frac{(a-x)^3}{2\,.\,3}\,d\,[f'''(x)] \\
&= af'(0) + \frac{a^2}{2} f''(0) + \frac{a^3}{2\,.\,3} f'''(0) + \int_0^a \frac{(a-x)^3}{2\,.\,3} f''''(x)\,dx.
\end{aligned}$$

Clearly we can repeat this process as often as we like, we then obtain

$$f(a) = f(0) + af'(0) + \frac{a^2}{\underline{|2}} f''(0) + \frac{a^3}{\underline{|3}} f'''(0) + \dots + \frac{a^n}{\underline{|n}} f^n(0) + \int_0^a \frac{(a-x)^n}{\underline{|n}} f^{n+1}(x)\,dx.$$

Whether this expansion is of any use depends on whether $\int_0^a \frac{(a-x)^n}{\lfloor n} f^{n+1}(x)\,dx$ decreases indefinitely as n increases.

Now when x varies from 0 to a, $(a-x)^n$ varies from a^n to 0; suppose that $f^{n+1}(x)$ lies between the numbers $+M$ and $-M$, then $\frac{(a-x)^n}{\lfloor n} f^{n+1}(x)$ lies between $\frac{Ma^n}{\lfloor n}$ and $-\frac{Ma^n}{\lfloor n}$.

$\therefore$ the integral lies between $\pm\frac{Ma^{n+1}}{\lfloor n}$.

Hence we can say that if we replace $f(a)$ by the terminating series

$$f(0)+af'(0)+\frac{a^2}{\lfloor 2}f''(0)+\ldots+\frac{a^n}{\lfloor n}f^n(0),$$

the error is less than $\frac{Ma^{n+1}}{\lfloor n}$, where M is the greatest value of $f^{n+1}(x)$ when x varies from 0 to a.

If this error can be made as small as we please by taking n sufficiently large, we say that

$$f(a)=f(0)+af'(0)+\frac{a^2}{\lfloor 2}f''(0)+\ldots$$
$$+\frac{a^n}{\lfloor n}f^n(0)+\frac{a^{n+1}}{\lfloor n+1}f^{n+1}(0)+\ldots\to\infty.$$

Example 7.

Expand $(z+a)^n$ where n is a positive integer.

Here $f(a)\equiv(z+a)^n$, $\quad\therefore f(0)=z^n$;

$f'(a)=n(z+a)^{n-1}$, $\quad\therefore f'(0)=nz^{n-1}$;

$f''(a)=n(n-1)(z+a)^{n-2}$, $\quad\therefore f''(0)=n(n-1)z^{n-2}$;

and so on $\quad$ ....

$f^n(a)=\lfloor n$, $\quad\therefore f^n(0)=\lfloor n$;

$f^{n+1}(a)=0$, $\quad\therefore \int_0^a\frac{(a-x)^n}{\lfloor n}f^{n+1}(x)\,dx=0.$

$$\therefore (z+a)^n=z^n+a\,.\,nz^{n-1}+\frac{a^2}{\lfloor 2}\,.\,n(n-1)z^{n-2}+\ldots\ldots+\frac{a^n}{\lfloor n}\,.\,\lfloor n+0$$
$$=z^n+nz^{n-1}a+\frac{n(n-1)}{\lfloor 2}z^{n-2}a^2+\ldots\ldots+a^n.$$

Maclaurin's Theorem can be written in a different form, known as Taylor's Theorem.

Taylor's Theorem

If as z varies from x to $x+h$, $f(z)$ possesses successive differential coefficients, then

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\ldots$$
$$+\frac{h^n}{\lfloor n}f^n(x)+\int_x^{x+h}\frac{(x+h-z)^n}{\lfloor n}f^{n+1}(z)\,dz.$$

Let $f(x+h)\equiv\phi(h)$ so that $f(x)=\phi(0)$.

Then

$$\phi'(h)=\frac{d}{dh}f(x+h)=\frac{d\,f(x+h)}{d\,(x+h)}\times\frac{d\,(x+h)}{dx}=f'(x+h),$$
$$\therefore\ \phi'(0)=f'(x).$$

Similarly $\phi''(h)=f''(x+h)$, $\phi''(0)=f''(x)$, etc.

By Maclaurin's Theorem

$$\phi(h)=\phi(0)+h\,\phi'(0)+\frac{h^2}{\lfloor 2}\phi''(0)+\ldots$$
$$+\frac{h^n}{\lfloor n}\phi^n(0)+\int_0^h\frac{(h-u)^n}{\lfloor n}\phi^{n+1}(u)\,du.$$

Now

$$\int_0^h\frac{(h-u)^n}{\lfloor n}\phi^{n+1}(u)\,du=\int_0^h\frac{(h-u)^n}{\lfloor n}f^{n+1}(x+u)\,du \qquad \text{put } x+u=z$$
$$=\int_x^{x+h}\frac{(h+x-z)^n}{\lfloor n}f^{n+1}(z)\,dz.$$

$$\therefore\ f(x+h)=\phi(h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\ldots$$
$$+\frac{h^n}{\lfloor n}f^n(x)+\int_x^{x+h}\frac{(h+x-z)^n}{\lfloor n}f^{n+1}(z)\,dz.$$

EXAMPLES XVII c

1. Show that $\sin a = a - \frac{a^3}{\lfloor 3} + \int_0^a \frac{(a-x)^3}{\lfloor 3} \sin x \, dx$,

and deduce that the error in replacing $\sin a$ by $a - \frac{a^3}{\lfloor 3}$ is less than $\frac{a^4}{6}$.

2. Show that $\log(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \int_0^a \frac{(a-x)^3}{(1+x)^4} dx$,

and deduce that the error in replacing $\log(1+a)$ by $a - \frac{a^2}{2} + \frac{a^3}{3}$ is less than a^4, if $0 < a < 1$.

3. Show that the error in replacing $\cos x$ by $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots + (-1)^n \frac{x^{2n}}{\lfloor 2n}$ is less than $\frac{x^{2n+1}}{\lfloor 2n+1}$.

4. If $f(x)$ and $f'(x)$ are continuous as x varies from a to $a+h$, use the relation $f(a+h) - f(a) = \int_a^{a+h} f'(x)\,dx$ to prove that there exists a number θ between 0 and 1 such that $f(a+h) - f(a) = hf'(a+\theta h)$.

5. By drawing the graph of $y = f(x)$, interpret geometrically the meaning of θ in the relation $\frac{f(a+h) - f(a)}{h} = f'(a+\theta h)$.

Indeterminate Forms

Example 8.

Find the limit of $\frac{e^x - e^{-x}}{x}$ when $x \to 0$.

If we put $x=0$ in $\frac{e^x - e^{-x}}{x}$ we obtain $\frac{1-1}{0}$ or $\frac{0}{0}$ which is meaningless: the function has therefore no meaning when $x=0$, but it has a definite value for every value of x other than $x=0$, and as $x \to 0$ the function may tend to a definite value.

$$\text{Now}\quad e^x - e^{-x} = \left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\right) - \left(1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots\right)$$

$$= 2\left(x + \frac{x^3}{\lfloor 3} + \ldots\right).$$

$$\therefore \frac{e^x - e^{-x}}{x} = 2\left(1 + \frac{x^2}{\lfloor 3} + \ldots\right) \text{ provided } x \neq 0.$$

$$\therefore \text{ as } x \to 0,\ \frac{e^x - e^{-x}}{x} \to 2 \text{ and we say } \underset{x\to 0}{\text{Lt}} \frac{e^x - e^{-x}}{x} = 2.$$

We shall now prove the following theorem:

If $f(x)$ and $\phi(x)$ are two functions which possess successive differential coefficients and such that when $x=a$, $f(a)=0=\phi(a)$, then

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{x\to a}\frac{f'(x)}{\phi'(x)}.$$

By hypothesis, $\dfrac{f(x)}{\phi(x)}$ has no meaning when $x=a$ but it may tend to a definite limit when $x\to a$.

By Taylor's Theorem

$$\frac{f(a+h)}{\phi(a+h)}=\frac{f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\dots}{\phi(a)+h\phi'(a)+\frac{h^2}{\lfloor 2}\phi''(a)+\dots}$$

$$=\frac{hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\dots}{h\phi'(a)+\frac{h^2}{\lfloor 2}\phi''(a)+\dots}\qquad\text{since } f(a)=0=\phi(a)$$

$$=\frac{f'(a)+\frac{h}{\lfloor 2}f''(a)+\dots}{\phi'(a)+\frac{h}{\lfloor 2}\phi''(a)+\dots}\quad\text{if } h\neq 0.$$

$$\therefore\ \text{when } h\to 0,\qquad \frac{f(a+h)}{\phi(a+h)}\to\frac{f'(a)}{\phi'(a)}.$$

$$\therefore\ \operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{h\to 0}\frac{f(a+h)}{\phi(a+h)}=\frac{f'(a)}{\phi'(a)}.$$

If however we also have

$$f'(a)=0=\phi'(a),$$

we obtain in the same way

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\frac{f''(a)}{\phi''(a)}$$

and so on.

Example 9.

Evaluate $$\underset{x\to\infty}{\text{Lt}}\, x\sin\left(\frac{1}{x}\right),$$

put $x=\frac{1}{u}$; when $x\to\infty$, $u\to 0$,

$$\therefore\ \underset{x\to\infty}{\text{Lt}}\, x\sin\left(\frac{1}{x}\right)=\underset{u\to 0}{\text{Lt}}\,\frac{1}{u}\sin u=\underset{u\to 0}{\text{Lt}}\,\frac{\sin u}{u}=1.$$

EXAMPLES XVII d

Evaluate the following limits:

1. $\underset{x\to 1}{\text{Lt}}\,\dfrac{x^5-x^4}{x^3+3x^2-4}$. **2.** $\underset{x\to 1}{\text{Lt}}\,\dfrac{\log x}{x-1}$. **3.** $\underset{x\to 0}{\text{Lt}}\,\dfrac{e^x-e^{-x}}{x}$.

4. $\underset{x\to 0}{\text{Lt}}\,\dfrac{1-\cos 2x}{1-\cos 3x}$. **5.** $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\,\dfrac{\frac{\pi}{2}-x}{\cos 3x}$. **6.** $\underset{x\to 0}{\text{Lt}}\,\sin x\log x$.

7. $\underset{x\to\infty}{\text{Lt}}\,x^2\left[1-\cos\left(\frac{1}{x}\right)\right]$. **8.** $\underset{x\to\infty}{\text{Lt}}\,(\sqrt{x^2+3x}-\sqrt{x^2+2x})$.

9. $\underset{x\to\infty}{\text{Lt}}\,\dfrac{x+\log x}{x\log x}$.

10. If $\cos 2n\theta\equiv a_0+a_1\sin^2\theta+a_2\sin^4\theta+\ldots+a_n\sin^{2n}\theta$, where n is an integer, evaluate a_0 by putting $\theta=0$ and evaluate a_1 by taking the limit when $\theta\to 0$ of the relation

$$\frac{\cos 2n\theta-a_0}{\sin^2\theta}\equiv a_1+a_2\sin^2\theta+\ldots.$$

11. If $\dfrac{\sin 2n\theta}{\sin\theta\cos\theta}\equiv a_0+a_1\sin^2\theta+a_2\sin^4\theta+\ldots+a_{n-1}\sin^{2n-2}\theta$, where n is an integer, evaluate a_0 and a_1 by the limit method indicated in Ex. 10.

MISCELLANEOUS EXAMPLES 32—35

M. 32

1. (i) If $p=c\,.\,e^{\frac{\theta}{\theta-a}}$ where c, a are constants, find $\dfrac{1}{p}\dfrac{dp}{d\theta}$ in terms of θ.

(ii) Expand $\log(\cosh x)$ in powers of x as far as x^4.

2. A particle of mass 1 lb. falls vertically under gravity and the air resistance is kv^2 lbs. when the velocity is v ft. sec.; using the equation $\dfrac{dv}{dt}=g(1-kv^2)$, prove that in t secs. it falls $\dfrac{1}{gk}\log[\cosh(gt\sqrt{k})]$ feet.

3. Find the length of the subtangent to the curve $y=\frac{x^3}{2a-x}$ at the point (a, a).

4. A shaft is rotating under a variable couple $a \sin^2 \theta$ where θ is the angle turned through at any time. Find the amount of work done per revolution.

5. If $x=f(t)$ and $y=\phi(t)$ and if $\frac{d^2y}{dx^2}=0$, prove that

$$\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} = \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}.$$

If a curve is given by $x=a \cos t+\frac{1}{2} b \cos 2t$, $y=a \sin t+\frac{1}{2} b \sin 2t$, prove that the points for which $\frac{d^2y}{dx^2}=0$ are given by $\cos t=-\frac{a^2+2b^2}{3ab}$. What geometrical condition is satisfied at these points?

M. 33

1. If $y^{\frac{1}{m}}=x+\sqrt{(1+x^2)}$, prove that $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=m^2y$.

2. (i) If $0<a<1$ show that $\int_0^a x \log(1+x)\,dx<\frac{a^3}{3}$

(ii) Prove that $\int_0^1 x \log\left(1+\frac{x}{2}\right) dx=\frac{3}{4}-\frac{3}{2}\log\left(\frac{3}{2}\right)$.

3. If the tangent at a point P of a curve cuts the x-axis OX at T and if for all positions of P the ordinate of P equals OT, find the equation of the curve.

4. A body has a circular base centre O, radius a, and the height at any point P of the base is $a \cos \frac{\pi x}{2a}$ where $OP=x$. Calculate the volume of the body and the height of its centre of gravity above the base.

5. A circle of radius a rolls without slipping on the outside of a fixed circle of radius $2a$, centre O; a point P on the rim of the moving circle is initially in contact with the fixed circle at B, OA is a radius perpendicular to OB; when the moving circle touches the fixed circle at Q, show that the tangent to the locus of P is perpendicular to QP and if p is the length of the perpendicular from O to this tangent and if the perpendicular makes with OA an angle ψ, prove that the locus of P is given by $p=4a \sin \frac{\psi}{2}$ or $p=4a \cos \frac{\psi}{2}$. Use the relation $r\frac{dr}{dp}=\rho=p+\frac{d^2p}{d\psi^2}$ to prove that the pedal equation of the locus of P is $4(r^2-4a^2)=3p^2$.

M. 34

1. A regular pyramid, vertex O, stands on a square base $ABCD$. Its net is formed by cutting down the edges OA, OB, OC, OD and folding the faces flat in the plane of the base. If the net had been obtained from a sheet of cardboard 2 feet square with its sides parallel to AB, BC, what length should AB have to give the maximum volume for the pyramid?

2. A cylinder 2″ in diameter (see Fig. 181) is provided with a groove whose section is a semicircle diameter $\frac{1}{2}$″; find the area of the surface of the groove.

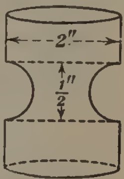

Fig. 181.

3. Find a point on the curve $2y=5+x^4$ such that the tangent at that point passes through the origin.

4. By using the formula $a^2=b^2+c^2-2bc\cos A$ for the triangle ABC, prove that if A increases by 1′ and if b, c remain constant, the increase in a is approximately $\frac{\pi p}{10800}$, where p is the perpendicular from A to BC.

5. Show that $y=x\cos 2x+2x^2\sin 2x$ is a solution of the equation $\frac{d^2y}{dx^2}+4y=16x\cos 2x$ and that $y=a\sin 2x+b\cos 2x+x\cos 2x+2x^2\sin 2x$ is the general solution.

M 35

1. ACB is a horizontal line and BE a vertical line: a rod CD is hinged at C and turns about C in a vertical plane at the uniform rate of 1 revolution per minute. BP is the shadow of CD cast by a light at A on the wall BE; $AC=10$ feet, $AB=24$ feet, $CD=5$ feet. Find the velocity of P when D is 3 feet above AC.

2. A sphere of radius a has a hollow concentric spherical cavity of radius $\frac{2a}{3}$. Show that a plane whose distance from the centre is $\frac{19a}{45}$ divides the sphere into two portions whose volumes are in the ratio 3 : 1.

3. Through the fixed point (a, b) a line is drawn so that the portion intercepted between the axes is a minimum; prove that its length is

$$(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}.$$

4. If $y=4x^2(1-x^2)$, find the square root of the mean value of y^2 for equal intervals of x between $x=0$ and $x=1$. Interpret the result geometrically, using the idea of a volume.

5. Find the ordinate of the point on the curve $y=a\sin\frac{x}{p}$ at which the centre of the circle of curvature lies on the x-axis.

LOGARITHMIC TABLES

NAPIERIAN LOGARITHMS (BASE e)

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
1·0	0·0000	0100	0198	0296	0392	0488	0583	0677	0770	0862	10	19	29	38	48	57	67	76	86
1·1	0·0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	9	17	26	35	44	52	61	70	78
1·2	0·1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	8	16	24	32	40	48	56	64	72
1·3	0·2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	7	15	22	30	37	44	52	59	67
1·4	0·3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	7	14	21	28	35	41	48	55	62
1·5	0·4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	7	13	19	26	32	39	45	52	58
1·6	0·4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	6	12	18	24	30	36	42	48	55
1·7	0·5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	6	11	17	23	29	34	40	46	52
1·8	0·5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	5	11	16	22	27	32	38	43	49
1·9	0·6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	5	10	15	20	26	31	36	41	46
2·0	0·6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	5	10	15	20	24	29	34	39	44
2·1	0·7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	5	9	14	19	23	28	33	37	42
2·2	0·7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	4	9	13	18	22	27	31	36	40
2·3	0·8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	4	9	13	17	21	26	30	34	38
2·4	0·8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	4	8	12	16	20	24	29	33	37
2·5	0·9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	4	8	12	16	20	24	27	31	35
2·6	0·9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	4	8	11	15	19	23	26	30	34
2·7	0·9933	9969	$\overline{0006}$	$\overline{0043}$	$\overline{0080}$	$\overline{0116}$	$\overline{0152}$	$\overline{0188}$	$\overline{0225}$	$\overline{0260}$	4	7	11	15	18	22	26	29	33
2·8	1·0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	4	7	11	14	18	21	25	28	32
2·9	1·0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	3	7	10	14	17	20	24	27	31
3·0	1·0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	3	7	10	13	16	20	23	26	30
3·1	1·1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	3	6	10	13	16	19	22	25	29
3·2	1·1[illegible]32	1663	1694	1725	1756	1787	1817	1848	1878	1909	3	6	9	12	15	18	21	25	28
3·3	1·1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	3	6	9	12	15	18	21	24	27
3·4	1·2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	3	6	9	12	14	17	20	23	26
3·5	1·2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	3	6	8	11	14	17	20	22	25
3·6	1·2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	3	5	8	11	14	16	19	22	25
3·7	1·3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	3	5	8	11	13	16	19	21	24
3·8	1·3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	3	5	8	10	13	16	18	21	23
3·9	1·3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	3	5	8	10	13	15	18	20	23
4·0	1·3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	[illegible]	5	7	10	12	15	17	20	22
4·1	1·4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	2	5	7	10	12	14	17	19	22
4·2	1·4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	2	5	7	9	12	14	16	19	21
4·3	1·4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	2	5	7	9	12	14	16	18	21
4·4	1·4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	[illegible]	4	7	[illegible]	11	13	16	18	20
4·5	1·5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	2	4	7	9	11	13	15	18	20
4·6	1·5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	2	4	6	9	11	13	15	17	19
4·7	1·5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	2	4	6	8	11	13	15	17	19
4·8	1·5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	[illegible]	4	6	8	10	12	14	16	19
4·9	1·5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	2	4	6	8	10	12	14	16	18
5·0	1·6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	2	4	6	8	10	12	14	16	18
5·1	1·6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	2	4	6	8	10	12	14	16	18
5·2	1·6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	2	4	6	8	10	12	13	15	17
5·3	1·6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	2	4	6	[illegible]	9	11	13	15	17
5·4	1·6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	2	4	6	7	9	11	13	15	17

NAPIERIAN LOGARITHMS OF 10^n

n	1	2	3	4	5	6	7	8	9	10
$\log_e 10^n$	2·3026	4·6052	6·9078	9·2103	11·5129	13·8155	16·1181	18·4207	20·7233	23·0259

NAPIERIAN LOGARITHMS (BASE *e*)

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5·5	1·7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	2	4	5	7	9	11	13	14	16
5·6	1·7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	2	4	5	7	9	11	12	14	16
5·7	1·7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	2	4	5	7	9	10	12	14	16
5·8	1·7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	2	3	5	7	9	10	12	14	15
5·9	1·7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	2	3	5	7	8	10	12	13	15
6·0	1·7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	2	3	5	7	8	10	12	13	15
6·1	1·8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	2	3	5	7	8	10	11	13	15
6·2	1·8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	2	3	5	6	8	10	11	13	14
6·3	1·8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	2	3	5	6	8	10	11	13	14
6·4	1·8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	2	3	5	6	8	9	11	12	14
6·5	1·8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	2	3	5	6	8	9	11	12	14
6·6	1·8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	2	3	5	6	8	9	11	12	14
6·7	1·9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	2	3	4	6	7	9	10	12	13
6·8	1·9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	2	3	4	6	7	9	10	12	13
6·9	1·9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	1	3	4	6	7	9	10	12	13
7·0	1·9459	9473	9488	9502	9516	9530	9544	9559	9573	9587	1	3	4	6	7	9	10	11	13
7·1	1·9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	1	3	4	6	7	8	10	11	13
7·2	1·9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	1	3	4	6	7	8	10	11	12
7·3	1·9879	9892	9906	9920	9933	9947	9961	9974	9988	$\overline{0001}$	1	3	4	5	7	8	10	11	12
7·4	2·0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	1	3	4	5	7	8	9	11	12
7·5	2·0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	1	3	4	5	7	8	9	11	12
7·6	2·0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	1	3	4	5	7	8	9	11	12
7·7	2·0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	1	3	4	5	7	8	9	10	12
7·8	2·0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	1	3	4	5	6	8	9	10	12
7·9	2·0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	1	3	4	5	6	8	9	10	11
8·0	2·0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	1	3	4	5	6	8	9	10	11
8·1	2·0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	1	3	4	5	6	7	9	10	11
8·2	2·1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	1	2	4	5	6	7	9	10	11
8·3	2·1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	1	2	4	5	6	7	8	10	11
8·4	2·1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	1	2	4	5	6	7	8	10	11
8·5	2·1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	1	2	4	5	6	7	8	9	11
8·6	2·1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	1	2	3	5	6	7	8	9	10
8·7	2·1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	1	2	3	5	6	7	8	9	10
8·8	2·1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	1	2	3	5	6	7	8	9	10
8·9	2·1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	1	2	3	5	6	7	8	9	10
9·0	2·1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	1	2	3	4	6	7	8	9	10
9·1	2·2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	1	2	3	4	6	7	8	9	10
9·2	2·2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	1	2	3	4	5	7	8	9	10
9·3	2·2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	1	2	3	4	5	6	8	9	10
9·4	2·2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	1	2	3	4	5	6	7	9	10
9·5	2·2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	1	2	3	4	5	6	7	8	10
9·6	2·2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	1	2	3	4	5	6	7	8	9
9·7	2·2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	1	2	3	4	5	6	7	8	9
9·8	2·2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	1	2	3	4	5	6	7	8	9
9·9	2·2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	1	2	3	4	5	6	7	8	9

NAPIERIAN LOGARITHMS OF 10^{-n}

n	1	2	3	4	5	6	7	8	9	10
$\log_e 10^{-n}$	$\overline{3}$·6974	$\overline{5}$·3948	$\overline{7}$·0922	$\overline{10}$·7897	$\overline{12}$·4871	$\overline{14}$·1845	$\overline{17}$·8819	$\overline{19}$·5793	$\overline{21}$·2767	$\overline{24}$·9741

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
·00	1·0000	1·0000	1·0000	·0000	·40	1·4918	·6703	1·0811	·4108
·01	1·0101	·9901	1·0001	·0100	·41	1·5068	·6636	1·0852	·4216
·02	1·0202	·9802	1·0002	·0200	·42	1·5220	·6570	1·0895	·4325
·03	1·0305	·9704	1·0005	·0300	·43	1·5373	·6505	1·0939	·4434
·04	1·0408	·9608	1·0008	·0400	·44	1·5527	·6440	1·0984	·4543
·05	1·0513	·9512	1·0012	·0500	·45	1·5683	·6376	1·1030	·4653
·06	1·0618	·9418	1·0018	·0600	·46	1·5841	·6313	1·1077	·4764
·07	1·0725	·9324	1·0025	·0701	·47	1·6000	·6250	1·1125	·4875
·08	1·0833	·9231	1·0032	·0801	·48	1·6161	·6188	1·1174	·4986
·09	1·0942	·9139	1·0040	·0901	·49	1·6323	·6126	1·1225	·5098
·10	1·1052	·9048	1·0050	·1002	·50	1·6487	·6065	1·1276	·5211
·11	1·1163	·8958	1·0061	·1102	·51	1·6653	·6005	1·1329	·5324
·12	1·1275	·8869	1·0072	·1203	·52	1·6820	·5945	1·1383	·5438
·13	1·1388	·8781	1·0085	·1304	·53	1·6989	·5886	1·1438	·5552
·14	1·1503	·8694	1·0098	·1405	·54	1·7160	·5827	1·1494	·5666
·15	1·1618	·8607	1·0113	·1506	·55	1·7333	·5769	1·1551	·5782
·16	1·1735	·8521	1·0128	·1607	·56	1·7507	·5712	1·1609	·5897
·17	1·1853	·8437	1·0145	·1708	·57	1·7683	·5655	1·1669	·6014
·18	1·1972	·8353	1·0162	·1810	·58	1·7860	·5599	1·1730	·6131
·19	1·2092	·8270	1·0181	·1911	·59	1·8040	·5543	1·1792	·6248
·20	1·2214	·8187	1·0201	·2013	·60	1·8221	·5488	1·1855	·6367
·21	1·2337	·8106	1·0221	·2115	·61	1·8404	·5434	1·1919	·6485
·22	1·2461	·8025	1·0243	·2218	·62	1·8589	·5379	1·1984	·6605
·23	1·2586	·7945	1·0266	·2320	·63	1·8776	·5326	1·2051	·6725
·24	1·2712	·7866	1·0289	·2423	·64	1·8965	·5273	1·2119	·6846
·25	1·2840	·7788	1·0314	·2526	·65	1·9155	·5220	1·2188	·6967
·26	1·2969	·7711	1·0340	·2629	·66	1·9348	·5169	1·2258	·7090
·27	1·3100	·7634	1·0367	·2733	·67	1·9542	·5117	1·2330	·7213
·28	1·3231	·7558	1·0395	·2837	·68	1·9739	·5066	1·2403	·7336
·29	1·3364	·7483	1·0423	·2941	·69	1·9937	·5016	1·2476	·7461
·30	1·3499	·7408	1·0453	·3045	·70	2·0138	·4966	1·2552	·7586
·31	1·3634	·7334	1·0484	·3150	·71	2·0340	·4916	1·2628	·7712
·32	1·3771	·7261	1·0516	·3255	·72	2·0544	·4868	1·2706	·7838
·33	1·3910	·7189	1·0549	·3360	·73	2·0751	·4819	1·2785	·7966
·34	1·4049	·7118	1·0584	·3466	·74	2·0959	·4771	1·2865	·8094
·35	1·4191	·7047	1·0619	·3572	·75	2·1170	·4724	1·2947	·8223
·36	1·4333	·6977	1·0655	·3678	·76	2·1383	·4677	1·3030	·8353
·37	1·4477	·6907	1·0692	·3785	·77	2·1598	·4630	1·3114	·8484
·38	1·4623	·6839	1·0731	·3892	·78	2·1815	·4584	1·3199	·8615
·39	1·4770	·6771	1·0770	·4000	·79	2·2034	·4538	1·3286	·8748
·40	1·4918	·6703	1·0811	·4108	·80	2·2255	·4493	1·3374	·8881

EXPONENTIAL AND HYPERBOLIC FUNCTIONS (*continued*)

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
·80	2·2255	·4493	1·3374	·8881	1·20	3·3201	·3012	1·8107	1·5095
·81	2·2479	·4449	1·3464	·9015	1·21	3·3535	·2982	1·8258	1·5276
·82	2·2705	·4404	1·3555	·9150	1·22	3·3872	·2952	1·8412	1·5460
·83	2·2933	·4360	1·3647	·9286	1·23	3·4212	·2923	1·8568	1·5645
·84	2·3164	·4317	1·3740	·9423	1·24	3·4556	·2894	1·8725	1·5831
·85	2·3397	·4274	1·3835	·9561	1·25	3·4903	·2865	1·8884	1·6019
·86	2·3632	·4232	1·3932	·9700	1·26	3·5254	·2837	1·9045	1·6209
·87	2·3869	·4190	1·4029	·9840	1·27	3·5609	·2808	1·9208	1·6400
·88	2·4109	·4148	1·4128	·9981	1·28	3·5966	·2780	1·9373	1·6593
·89	2·4351	·4107	1·4229	1·0122	1·29	3·6328	·2753	1·9540	1·6788
·90	2·4596	·4066	1·4331	1·0265	1·30	3·6693	·2725	1·9709	1·6984
·91	2·4843	·4025	1·4434	1·0409	1·31	3·7062	·2698	1·9880	1·7182
·92	2·5093	·3985	1·4539	1·0554	1·32	3·7434	·2671	2·0053	1·7381
·93	2·5345	·3945	1·4645	1·0700	1·33	3·7810	·2645	2·0228	1·7583
·94	2·5600	·3906	1·4753	1·0847	1·34	3·8190	·2618	2·0404	1·7786
·95	2·5857	·3867	1·4862	1·0995	1·35	3·8574	·2592	2·0583	1·7991
·96	2·6117	·3829	1·4973	1·1144	1·36	3·8962	·2567	2·0764	1·8198
·97	2·6379	·3791	1·5085	1·1294	1·37	3·9354	·2541	2·0947	1·8406
·98	2·6645	·3753	1·5199	1·1446	1·38	3·9749	·2516	2·1132	1·8617
·99	2·6912	·3716	1·5314	1·1598	1·39	4·0149	·2491	2·1320	1·8829
1·00	2·7183	·3679	1·5431	1·1752	1·40	4·0552	·2466	2·1509	1·9043
1·01	2·7456	·3642	1·5549	1·1907	1·41	4·0960	·2441	2·1701	1·9259
1·02	2·7732	·3606	1·5669	1·2063	1·42	4·1371	·2417	2·1894	1·9477
1·03	2·8011	·3570	1·5790	1·2220	1·43	4·1787	·2393	2·2090	1·9697
1·04	2·8292	·3535	1·5913	1·2379	1·44	4·2207	·2369	2·2288	1·9919
1·05	2·8577	·3499	1·6038	1·2539	1·45	4·2631	·2346	2·2488	2·0143
1·06	2·8864	·3465	1·6164	1·2700	1·46	4·3060	·2322	2·2691	2·0369
1·07	2·9154	·3430	1·6292	1·2862	1·47	4·3492	·2299	2·2896	2·0597
1·08	2·9447	·3396	1·6421	1·3025	1·48	4·3929	·2276	2·3103	2·0827
1·09	2·9743	·3362	1·6553	1·3190	1·49	4·4371	·2254	2·3312	2·1059
1·10	3·0042	·3329	1·6685	1·3357	1·50	4·4817	·2231	2·3524	2·1293
1·11	3·0344	·3296	1·6820	1·3524	1·51	4·5267	·2209	2·3738	2·1529
1·12	3·0649	·3263	1·6956	1·3693	1·52	4·5722	·2187	2·3955	2·1768
1·13	3·0957	·3230	1·7093	1·3863	1·53	4·6182	·2165	2·4174	2·2008
1·14	3·1268	·3198	1·7233	1·4035	1·54	4·6646	·2144	2·4395	2·2251
1·15	3·1582	·3166	1·7374	1·4208	1·55	4·7115	·2122	2·4619	2·2496
1·16	3·1899	·3135	1·7517	1·4382	1·56	4·7588	·2101	2·4845	2·2743
1·17	3·2220	·3104	1·7662	1·4558	1·57	4·8066	·2080	2·5073	2·2993
1·18	3·2544	·3073	1·7808	1·4735	1·58	4·8550	·2060	2·5305	2·3245
1·19	3·2871	·3042	1·7957	1·4914	1·59	4·9037	·2039	2·5538	2·3499
1·20	3·3201	·3012	1·8107	1·5095	1·60	4·9530	·2019	2·5775	2·3756

[P. T. O.

EXPONENTIAL AND HYPERBOLIC FUNCTIONS (*continued*)

x	e^x	e^{-x}	Cosh x	Sinh x	x	e^x	e^{-x}	Cosh x	Sinh x
1·60	4·9530	·2019	2·5775	2·3756	2·0	7·3891	·1353	3·7622	3·6269
1·61	5·0028	·1999	2·6013	2·4015	2·1	8·1662	·1225	4·1443	4·0219
1·62	5·0531	·1979	2·6255	2·4276	2·2	9·0250	·1108	4·5679	4·4571
1·63	5·1039	·1959	2·6499	2·4540	2·3	9·9742	·1003	5·0372	4·9370
1·64	5·1552	·1940	2·6746	2·4806	2·4	11·023	·0907	5·5570	5·4662
1·65	5·2070	·1921	2·6995	2·5075	2·5	12·183	·0821	6·1323	6·0502
1·66	5·2593	·1901	2·7247	2·5346	2·6	13·464	·0743	6·7690	6·6947
1·67	5·3122	·1882	2·7502	2·5620	2·7	14·880	·0672	7·4735	7·4063
1·68	5·3656	·1864	2·7760	2·5896	2·8	16·445	·0608	8·2527	8·1919
1·69	5·4195	·1845	2·8020	2·6175	2·9	18·174	·0550	9·1146	9·0596
1·70	5·4739	·1827	2·8283	2·6456	3·0	20·086	·0498	10·068	10·018
1·71	5·5290	·1809	2·8549	2·6740	3·1	22·198	·0450	11·122	11·076
1·72	5·5845	·1791	2·8818	2·7027	3·2	24·533	·0408	12·287	12·246
1·73	5·6407	·1773	2·9090	2·7317	3·3	27·113	·0369	13·575	13·538
1·74	5·6973	·1755	2·9364	2·7609	3·4	29·964	·0334	14·999	14·965
1·75	5·7546	·1738	2·9642	2·7904	3·5	33·116	·0302	16·573	16·543
1·76	5·8124	·1720	2·9922	2·8202	3·6	36·598	·0273	18·313	18·285
1·77	5·8709	·1703	3·0206	2·8503	3·7	40·447	·0247	20·236	20·211
1·78	5·9299	·1686	3·0493	2·8806	3·8	44·701	·0224	22·362	22·339
1·79	5·9895	·1670	3·0782	2·9112	3·9	49·402	·0202	24·711	24·691
1·80	6·0496	·1653	3·1075	2·9422	4·0	54·598	·0183	27·308	27·290
1·81	6·1104	·1637	3·1370	2·9734	4·1	60·340	·0166	30·178	30·162
1·82	6·1719	·1620	3·1669	3·0049	4·2	66·686	·0150	33·351	33·336
1·83	6·2339	·1604	3·1971	3·0367	4·3	73·700	·0136	36·857	36·843
1·84	6·2965	·1588	3·2277	3·0689	4·4	81·451	·0123	40·732	40·719
1·85	6·3598	·1572	3·2585	3·1013	4·5	90·017	·0111	45·014	45·003
1·86	6·4237	·1557	3·2897	3·1340	4·6	99·484	·0101	49·747	49·737
1·87	6·4883	·1541	3·3212	3·1671	4·7	109·95	·0091	54·978	54·969
1·88	6·5535	·1526	3·3530	3·2005	4·8	121·51	·0082	60·759	60·751
1·89	6·6194	·1511	3·3852	3·2342	4·9	134·29	·0074	67·149	67·141
1·90	6·6859	·1496	3·4177	3·2682	5·0	148·41	·0067	74·210	74·203
1·91	6·7531	·1481	3·4506	3·3025	5·1	164·02	·0061	82·014	82·008
1·92	6·8210	·1466	3·4838	3·3372	5·2	181·27	·0055	90·639	90·633
1·93	6·8895	·1451	3·5173	3·3722	5·3	200·34	·0050	100·17	100·17
1·94	6·9588	·1437	3·5512	3·4075	5·4	221·41	·0045	110·71	110·70
1·95	7·0287	·1423	3·5855	3·4432	5·5	244·69	·0041	122·35	122·34
1·96	7·0993	·1409	3·6201	3·4792	5·6	270·43	·0037	135·21	135·21
1·97	7·1707	·1395	3·6551	3·5156	5·7	298·87	·0033	149·44	149·43
1·98	7·2427	·1381	3·6904	3·5523	5·8	330·30	·0030	165·15	165·15
1·99	7·3155	·1367	3·7261	3·5894	5·9	365·04	·0027	182·52	182·52
2·00	7·3891	·1353	3·7622	3·6269	6·0	403·43	·0025	201·72	201·71

ANSWERS

PART I

EXAMPLES I a (p. 3)

1. (i) 3·6; (ii) 0; (iii) 1·8; (iv) -15; (v) 3, -5; (vi) (5, 6), (-7, $-1·2$); (vii) $3+1·2t$; (viii) $\frac{5a}{3}$.

2. (i) 1·2; (ii) 5; (iii) $2\frac{1}{2}$, 2; (iv) (1, 1·2), (-5, 6); (v) $2-\frac{4a}{5}$; (vi) $2-\frac{4}{5}(x+h)$; (vii) $\frac{6}{5}(1-t)$.

3. $3x-4y+12=0$; -4; 36° 52′; yes. **4.** $x^2+y^2=25$; yes; $\sqrt{(25-4a^2)}$.

5. 4, 9, 5, 68° 12′; $\frac{1}{4}a^2$, $\frac{1}{4}(a+h)^2$, $\frac{1}{4}(2ah+h^2)$, $\frac{1}{4}(2a+h)$; ± 3; yes.

6. (i) 2; (ii) -4; (iii) 1, 4, -4; (iv) 6 or -1; (v) (3, 2), (-2, -18); (vi) $(a-1)(4-a)$; (vii) $(a+h-1)(4-a-h)$.

9. (iii) 3; (iv) ± 3. **10.** (1, 1), (-1, -1); $a=2$.

11. (ii) $y=1, 1, -2, -1·5$; $x=-5$, 1·5 or 4·5, -1 or -3 or 7, 8·5.

EXAMPLES I b (p. 8)

1. $-2, -2, -3, 5, 4y^2-3, (a+b)^2-3$.

2. $0, 0, -\frac{1}{4}, 9t^2-3t, x^4-x^2, x^2+x, x^2+x$.

3. a^2+2, $(x+a)^2-2(x+a)+3$, $\frac{1}{x^2}(1-2x+3x^2)$. **4.** 10, 0·1.

6. $2x-2+h$, $2x-2$. **7.** 2, -3. **8.** 0, 1. **9.** $-\frac{1}{x^2+xh}$, $-\frac{1}{x^2}$.

10. 3, 5. **11.** less, more. **12.** -4, -2.

EXAMPLES I c (p. 11)

1. (1) $x>1$, $x<-3$; $x>3$, $x<-3$; $\frac{3}{2}>x>-2$: (2) $1>x>-3$; $3>x>2$ and $2>x>-3$; $x>\frac{3}{2}$, $x<-2$. **7.** $f(3), f(a)$; $f(x), f(x+h)$; $f(0)$.

8. $f(x+h)-f(x)$. **9.** 0, 0, 0; $\phi(1)$. **12.** 5, -5, -5, 5, 0, 5.

EXAMPLES II a (p. 16)

1. 0·6, 0·9, 0·6, 0·6; $\frac{3h}{5}$, $\frac{3k}{5}$, 0·6, 0·6; 30° 58′. **2.** $-0·6$, $-0·6$, 149° 2′.

3. 2, $\frac{1}{3}$. **4.** $-0·5$, $-0·5$. **5.** $\delta y=3\delta x$, 3.

6. 4, 10, 4, 5; $2ah+h+h^2$, $2ak+k+k^2$, $2a+1+h$, $2a+1+k$; $\delta x(2x+1+\delta x)$, $2x+1+\delta x$. **7.** 0, 0·9, 0·999; -2, $-1·1$, $-1·001$; $3-2x-\delta x$.

8. $f(x)$, $f(x+\delta x)$, tan QPR. **10.** 1·73, 0·27, $-0·27$, $-1·73$.

11. $\delta A=\delta x\,(2x+\delta x)$. **12.** $\delta s=16\delta t\,(2t+\delta t)$; 6·56; 65·6 ft. sec.

13. $\delta v=-\frac{500\delta p}{p\,(p+\delta p)}$; $-\frac{10}{11}$ cu. in.

14. $\delta R=\frac{1}{10^7}[3v^2\,\delta v+3v\,(\delta v)^2+(\delta v)^3]$; $-27·1$ lbs.

15. 1·3 sq. in. **16.** 380 cu. cms.

EXAMPLES II b (p. 21)

1. $x > 1{,}000{,}000$; 1. **2.** $1{\cdot}0005 > x > 0{\cdot}9995$; 2.
3. 20,001; 3. **4.** (ii) 2. **5.** (ii) 6.
6. $\delta y = 0{\cdot}42,\ 0{\cdot}0402,\ 0{\cdot}004002$; $\dfrac{\delta y}{\delta x} = 4{\cdot}2,\ 4{\cdot}02,\ 4{\cdot}002$; 4.
7. $5\delta x\,(2x+\delta x)$; $10x + 5\delta x$; $10x$. **8.** $3x^2$. **9.** $36 + 3h^2$; 36.
10. $3(a+h) - (a+h)^2$, $3a - a^2$, $3 - 2a - h$; $3 - 2a$. **11.** $-\dfrac{12}{x(x+\delta x)}$, $-\dfrac{12}{x^2}$.
12. $32t$. **13.** $A = \frac{1}{4}x^2$; $\delta A = \frac{1}{4}\delta x\,(2x+\delta x)$; $\frac{1}{2}x = PN$.
14. $\frac{1}{2}\{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3\}$; $\dfrac{3x^2}{2}$.
15. $10\frac{1}{2}$, 210; $\frac{1}{2}(n+1)$, $\frac{1}{2}n(n+1)$; $\frac{1}{2}$. **16.** $\dfrac{1+2+3+\ldots+n}{n^2} = \dfrac{n+1}{2n}$; $\frac{1}{2}$.
17. $\frac{1}{3}$; $\dfrac{(n-1)(2n-1)}{6n^2}$; $\frac{1}{3}$; $\frac{1}{3}$; $\dfrac{(n+1)(2n+1)}{6n^2}$; $\frac{1}{3}$. **19.** $2\pi r$; $4\pi r^2$.

EXAMPLES III a (p. 32)

1. (i) $4\delta x$, 4, 4; (ii) $5\delta x\,(2x+\delta x)$, $5(2x+\delta x)$, $10x$; (iii) $\delta x\,(2x+\delta x-3)$, $2x+\delta x-3$, $2x-3$; (iv) $-\dfrac{5\delta x}{x^2+x\,\delta x}$, $-\dfrac{5}{x^2+x\,\delta x}$, $-\dfrac{5}{x^2}$; (v) $\delta x\,(2x+\delta x+6)$, $2x+\delta x+6$, $2x+6$.
2. 8; 4, 0, -2; $-\frac{1}{3}$. **3.** $63°26'$, $45°$. **4.** $4x^3$. **5.** $21x^6$.
6. $6x^2$. **7.** $10x+4$. **8.** $16x^7-8$. **9.** $18x$. **10.** -1.
11. $1-3x^2$. **12.** px^{p-1}. **13.** $2qx^{2q-1}$. **14.** $\frac{3}{2}x$. **15.** $0{\cdot}1 - 1{\cdot}6x^3$.
16. $8x-4$. **17.** $3ax^2$. **18.** b. **19.** $3-2x$.
20. $2\pi x$. **21.** $4\pi x^2$. **22.** $75x^4$. **23.** $3x^3 - x$.
24. $5+6x+3x^2$. **25.** $2 - \dfrac{3}{x^2}$. **26.** $2x + \dfrac{1}{x^2}$.
27. $-\dfrac{1}{2x^2} - 3x^2$. **28.** $8t$; $3+32t$; $u+gt$.
29. 3. **30.** 0. **31.** $(1, -2)$, $(-1, 2)$.
32. $(1, 0)$, $(\frac{1}{3}, \frac{4}{27})$. **33.** 2, -1, 2.
34. $\frac{1}{2}x^2+c$; $x+c$; $\dfrac{3x^2}{2}+c$; $\frac{1}{4}x^4+c$; $\frac{1}{2}x^4+c$; x^5+c; $x-\frac{1}{2}x^3+c$; x^3+2x+c; πx^2+c; $\frac{4}{3}\pi x^3+c$; $\dfrac{1}{x}+c$; $-\dfrac{2}{x}+c$. **35.** $6t+4$. **36.** $ut+\frac{1}{2}at^2$.
43. $h+6$; 6. **44.** $10x^9$. **45.** na^{n-1}; nx^{n-1}.
46. $(1, -1)$, $(-1, 1)$. **47.** $12x^3$; $12x-14$; 12. **48.** -32; f.
49. 0. **51.** $\frac{1}{3}$, $\frac{125}{64}$. **52.** 2; 12, -12.

EXAMPLES III b (p. 36)

1. $11x - y = 5$. 2. $\frac{1}{3}, \frac{8}{9}, 9x + 9y = 11$. 3. $3x - y = 3, 3x + y + 6 = 0$.
4. $(1, 2)$; $3x - y = 1$. 5. $2y - 3x = 9, 8x + 12y = 67$. 6. $y = 6\frac{1}{27}, y = 7\frac{5}{8}$.
7. 0·8, 20. 8. $40x - y = 80$, or $y = 0$. 9. 10. 10. $\frac{5}{6}, \frac{5}{8}$.

EXAMPLES IV (p. 45)

1. +, −, −; max.; −2. 2. +, +; no; 0.
3. $OA+$, $ABC-$, $CDE+$, zero at A, C; AB, $BCD-$, zero at B; $AO-$, $OBC+$, $CDE-$, $EF+$, zero at O, C, E.
6. +, −, −, +; $\frac{4}{3}$ max., 6 min.; $\frac{11}{3}$.
7. (i) (1, 1) max.; (ii) (3, 2) min.; (iii) (1, −2) min., (−1, 2) max.; (iv) $(-\frac{1}{2}, -\frac{1}{4})$ min., (0, 0) max., $(\frac{1}{2}, -\frac{1}{4})$ min.; (v) (3, −27) min.; (vi) (−2, −36) min., (0, −20) max., (2, −36) min.; (vii) (0, 20) min.; (viii) none; (ix) (2, 12) min.
8. 100 ft. 9. 11,250 sq. yds. 10. 1·6 in. 11. 50 ft. 12. 2 cu. ft.
13. 6·53 ft. wide, 7·35 ft. high. 14. 3. 15. $\frac{4}{3}$ cu. ft.
16. $\frac{2}{3}$. 17. 4·5 ins. 18. 3, $x = 1$. 19. $7\frac{1}{27}$ cu. ft.
20. $\sqrt{\left(\frac{nR}{r}\right)}$; $\frac{1}{2}E\sqrt{\left(\frac{n}{Rr}\right)}$. 22. $2\frac{1}{4}$. 23. $\frac{4}{5}$.
25. $\frac{R_2}{R_1}$; $\frac{I^2 R_1 R_2}{R_1 + R_2}$. 26. 8 ins. apart. 27. $\frac{1}{3}AB$ from B.
28. 48° 22′ to 53° 58′. 29. $AP = \frac{2}{3}AB$. 30. $\frac{4}{9}$.
31. $\frac{1}{4}(5x^2 + 80)$ lbs., $\frac{5x}{2} + \frac{40}{x}$ lbs., $x = 4$.

EXAMPLES V a (p. 57)

1. $16(2t + \delta t)$, 48 ft. sec. 2. 4 sq. cms. per sec., 4·2 sq. cms.
3. 18, −14 ft. sec. 4. 11° 19′, 5° 43′, 0°, −11° 19′.
7. $R_0 \delta\theta(\alpha + 2\beta\theta)$. 8. $-6 \cdot 7 \times 10^{-5}$.
9. Distance decreases about 14 feet per degree increase of depression.
16. $\frac{dA}{dr} = kr$. 17. $\frac{dy}{dx} = C\frac{dy}{dz}$. 18. $\frac{dx}{dt} = kx$. 19. $\frac{dx}{dt} = -k\sqrt{x}$.
20. $\frac{dv}{dt} = kv^2$. 21. $\frac{d\theta}{dt} = k\theta$. 22. $\frac{dN}{dt} = -kN^2$. 23. $\frac{dT}{dt} = -k(T - C)$.
24. $\frac{dy}{dx} = kx$. 25. 0·50 cu. in. 26. $1 \cdot 5(\delta x)^2$. 27. $3h^2 + h^3$.
28. 207; 101,000. 29. $12(2z - 1)\delta z$. 30. $3(z^2 - 7z - 3)^2(2z - 7)\delta z$.

EXAMPLES V b (p. 63)

1. $\frac{1}{3}x^{-\frac{2}{3}}$; $\frac{3}{4}x^{-\frac{1}{4}}$; $-2x^{-3}$; $-4x^{-5}$; $-\frac{1}{2}x^{-\frac{3}{2}}$. 2. $-\frac{1\cdot4p}{v}$. 3. $-\frac{2K}{x^3}$, $\frac{6K}{x^4}$.
4. $\frac{2}{r}$; $\frac{4}{3}kV^{-\frac{1}{3}}$. 5. $-\frac{2}{(2x+3)^2}$; $\frac{2x}{(4-x^2)^2}$; $\frac{6}{(5-3x)^3}$; $\frac{5(2x-3)}{(x-1)^2(x-2)^2}$.
6. $\frac{1}{2\sqrt{(1+x)}}$; $-\frac{x}{\sqrt{(1-x^2)}}$; $\frac{2}{\sqrt{(3+4x)}}$; $-3x\sqrt{(1-x^2)}$; $\frac{2x}{(1-x^2)^2}$; $\frac{1}{3}(2x-3)(x^2-3x+7)^{-\frac{2}{3}}$.
8. $\frac{2}{3}x^{\frac{3}{2}}+c$; x^3+c; $2\sqrt{x}+c$; $c-\frac{1}{t}$; $\frac{2}{5}t^{\frac{5}{2}}+t^{\frac{2}{3}}+c$.

EXAMPLES V c (p. 66)

1. 10·5 cu. ft. per min. 2. 12·5 min. 3. 0·9 cu. cm. per min.
4. 6 sq. ins. per in. 5. 0·63 sq. in. per sec.
6. 120 lbs. per sq. in. decrease per min. 7. 8; 10 sec.
8. $\frac{1}{4}$ in. per sec.; $\frac{1}{h^2(3-h)}$ in. per sec.; glass tapers to a point at $h=3$.
9. 2·58 ins. per min. 10. 0·038 in. per sec. 11. 4 sq. ins. per min.
12. 2·3 ins. per min. 13. 32 sq. ins. 14. 1 : 4.
15. $\frac{1}{10\pi}(15-r)$. 18. 0·153 sq. in. 19. 216, 0, -8 ft. per sec.
20. $\frac{2a(a-b\cos C)}{c^2}$ per cent.

REVISION PAPERS R. 1—5 (p. 68)

R. 1. 1. $x=\pm1$, $y=-1$. 2. -5, $-3\cdot2$, $-3\cdot002$; -3.
3. $-\frac{200\,\delta v}{v(v+\delta v)}$; $-\frac{10}{21}=-0\cdot48$. 4. 5·99, 6·01; 6. 5. 2.
6. $\frac{1}{3}\pi x^2$; $\frac{\pi}{9}[3x^2\,\delta x+3x(\delta x)^2+(\delta x)^3]$, $\frac{1}{3}\pi x^2$.

R. 2. 1. $2x+3$. 2. $-3+6x-3x^2$; $3+4x+3x^2$; $1-\frac{2}{x^2}$.
3. (0, 10), (4, -54). 4. $x+2y=3$, $2x-y=1$. 6. 32.

R. 3. 1. 14·5. 3. -1; $3+\frac{1}{x^2}$; $36x^3-12x$. 4. $\frac{1}{6}$ cu. ft.
5. $\frac{4\,\delta x}{(1-x)^2}$. 6. $-\frac{3}{5}$, $-\frac{5}{3}$.

R. 4. 1. $1-\frac{4}{x^2}$; $\frac{3}{4}$, 0, -399, -3, 0, $\frac{3}{4}$. 2. $x=2$.
3. $(\frac{3}{2}, -22)$ min.; $(-\frac{3}{2}, 32)$ max. 4. $3\frac{1}{3}$ ins. from top.
5. 11 ft. 6. $(-1, 0)$ lies on curve; inflexion at (1, 14).

R. 5. 1. $\frac{dy}{dx}=\frac{y}{x}$; $\frac{dP}{dt}=\frac{Pr}{100}$.
3. Stationary at $x=0$, $1\frac{1}{4}$, 4; inflexions at $x=0\cdot57$, $2\cdot93$. 4. 2·60.
6. $\frac{3}{4}x^4-\frac{4}{5}x^5+c$; $\frac{1}{3}x^3+\frac{1}{x}+c$; $3x^3-6x^2+4x+c$; $\frac{1}{3}(2x)^{\frac{3}{2}}+c$.

MISCELLANEOUS EXAMPLES M. 1—6 (p. 71)

M. 1. 1. $s=\frac{1}{2}t^2$, 3 ft. sec. 2. $4\pi cr^2$ cu. ft. per sec. 3. 10 ins.
4. 2, -1, 2. 5. 2 per cent.

M. 2. 1. 0·3 in. 2. 80 ft. secs. 3. $\dfrac{kt}{\pi rh}$.
4. 3·1, 12·6, 28·3, 50·3 sq. ins.; conical with vertical angle 90°.
5. $2sh-2yk=k^2-h^2$; $\dfrac{s}{y}$; $\dfrac{s}{y}$.

M. 3. 1. 45°, 41° 11′ with horizontal. 2. 1·5 secs., -1·5 ft. secs.
3. $\sqrt[4]{\left(\dfrac{aW^2}{3b}\right)}$ ft. sec. 4. $p\,\delta v+v\,\delta p=B\,\delta t$; 0·464; 8·1 deg. per min.
5. $c=150{,}000$; 1° 9′.

M. 4. 1. 93·8, 86·6, 66·9 cms.; 1·92 cms. per hour.
2. $AC=r$; 22 sq. cms. per sec. 3. 0·0177, 0·0199 ins. per sec.
4. 50 ft. secs.; 119 ft. secs. 5. $-2<x<2$.

M. 5. 1. $(1, -4)$, $(9, 12)$; $2x+y+2=0$, $2x-3y+18=0$. 2. 16 ins. per sec.
3. 4 ins. per min. 4. $\frac{2}{3}$ ins. per sec.
5. $xy_1+yx_1=2c$; $(2x_1, 0)$, $(0, 2y_1)$; area $=2c$.

M. 6. 1. 2. 2. 15° 42′; 3600 ft. 3. -14 ft. sec. 4. 3·27.
5. 480 sq. ins.; 0·00823 in. per sec.

EXAMPLES VI (p. 79)

1. $y=4x^3+c$. 2. $y=x-\dfrac{1}{x}+c$. 3. $y=2\sqrt{x}+1$.

4. $y=\frac{1}{3}(x^3+2)$; $y=\frac{1}{4}x^4-2x^2+2$; $y=\frac{4}{3}x^3-2x^2+x+\frac{16}{3}$.

5. (i) $\frac{1}{2}x^2+c$; (ii) $\frac{1}{3}x^3+2x+c$; (iii) $\frac{7}{3}x^3+c$; (iv) x^4+c; (v) $\frac{1}{3}x^6+c$; (vi) $\frac{7}{11}x^{11}+c$; (vii) $\frac{5}{3}x^3-\frac{3}{2}x^2+8x+c$; (viii) $\frac{16}{3}x^3+4x^2+x+c$; (ix) $\frac{1}{3}x^3+\frac{1}{2}x^2-2x+c$; (x) $c-\dfrac{1}{x}$; (xi) $c-\dfrac{1}{2x^2}$; (xii) $c-\dfrac{5}{3x^3}$; (xiii) $c-\dfrac{7}{9x^9}$;

(xiv) $\frac{1}{3}x^3-\dfrac{1}{x}+c$; (xv) $\frac{2}{3}x^{\frac{3}{2}}+c$; (xvi) $\frac{3}{4}x^{\frac{4}{3}}+c$; (xvii) $2x^{\frac{1}{2}}+c$; (xviii) $\frac{3}{2}x^{\frac{2}{3}}+c$; (xix) $2x^{\frac{5}{2}}+c$; (xx) $\frac{1}{33}(3x-9)^{11}+c$; (xxi) $\frac{2}{21}(7x+1)^{\frac{3}{2}}+c$; (xxii) $c-\dfrac{1}{10(5x-2)^2}$.

6. (i) $\frac{1}{3}x^3-\frac{3}{2}x^2+4x+c$; (ii) $c-\dfrac{1}{4x^4}$; (iii) $\frac{2}{5}x^{\frac{5}{2}}+c$; (iv) $c-\frac{10}{3}x^{-0\cdot3}$; (v) $c-\dfrac{2}{x}+\dfrac{1}{2x^2}$; (vi) $\frac{10}{3}x^{0\cdot3}+c$; (vii) $3x+c$; (viii) $\frac{1}{24}(4x-1)^6+c$; (ix) $c-\dfrac{3}{x}-\frac{8}{3}x^{\frac{3}{2}}$.

7. $y=4x^3-7x+3$; $x=-\frac{3}{2}$, $x=\frac{1}{2}$. 8. 16 ft., 36 ft.

9. 35 lbs. 10. $4\frac{1}{2}$. 11. 3 ins. per sec.; 8 ins. 12. $\frac{1}{25}$, $1\frac{1}{2}$ lbs.

13. $y=5x^3+ax+b$. 14. $y=\frac{1}{12}x^4+ax+b$; $y=\dfrac{1}{2x}+ax+b$.

15. $y=\frac{1}{2}(x^3-x+4)$. 16. $\frac{1}{y^2}$; $x=c-\frac{1}{y}$. 17. $x=2\sqrt{y}-1$.
18. 0; $y=0\cdot0005\,(\frac{1}{12}x^4-2x^3+144x)$; $4^\circ\,7'$; 3·24 ins. 19. 360 ft. lbs.
20. $x=30t$, $y=16t^2$; $82\frac{1}{2}$ ft., $2\frac{3}{4}$ secs.; $64^\circ\,53'$ with horizontal.
21. 11·1 ft. secs. 22. 32 in./sec. 23. $\rho=\frac{0\cdot12}{\theta+15}$. 24. 10 ft. 25. 1:3.

EXAMPLES VII a (p. 85)

1. $(b-a)(b+a+3)$. 2. 9. 3. 21. 4. 15. 5. $3\frac{3}{4}$. 6. $5\frac{1}{3}$.

EXAMPLES VII b (p. 88)

1. (i) 2; (ii) $2\frac{1}{3}$; (iii) $11\frac{1}{3}$; (iv) $\frac{1}{2}$; (v) $2\frac{2}{3}$; (vi) $4\frac{2}{3}$; (vii) $-11\frac{2}{3}$; (viii) 12.
2. $2ah+\frac{2}{3}ch^3$; $\frac{2}{3}$; $\frac{1}{2}\pi c^4$; 1·205. 3. $2\frac{2}{3}$, $5\frac{1}{3}$, $5\frac{1}{3}$. 4. $4\frac{2}{3}$. 5. $4\frac{1}{2}$.
6. $10\frac{2}{3}$; 0. 7. $\frac{1}{4}a^4$; 1:3. 8. $-1\frac{1}{3}$. 9. 2. 10. $10\frac{1}{3}$.
11. $\frac{1}{4}$, $-\frac{1}{4}$; $11\frac{1}{4}$, $-\frac{7}{17}$. 12. $21\frac{3}{4}$. 13. $\frac{13}{12}$.
14. (2, 4), (−1, 4); $6\frac{3}{4}$; $6\frac{3}{4}$. 15. −57·15. 16. 12. 17. $\frac{9}{16}$.
18. $2\frac{2}{3}$. 19. $\frac{15}{16}$; 2, $\frac{7}{5}$ cu. ft. 20. $-2\frac{1}{3}$; $1\frac{2}{3}$; 9; $-2\frac{1}{3}$. 21. $17\frac{1}{15}$.
22. 25·6. 23. $\frac{2}{3}$. 24. 0·108.

EXAMPLES VII c (p. 93)

1. 524. 2. 6·28; 1·26. 3. 75·4. 4. 12·5 ins.; $\frac{625\pi}{3}=654$ cu. ins.
5. $\frac{64\pi}{3}=67\cdot0$. 6. $\frac{4}{3}\pi(b^2-a^2)^{\frac{3}{2}}$. 7. $3\cdot68\pi=11\cdot6$ cu. ft.
8. $22\cdot13\pi=69\cdot5$ cu. ins. 9. 30; $0\cdot1215\pi=0\,382$ cu. cm.
10. 1067 cu. ft. 11. 6070 cu. ins. 12. $\frac{1}{2}\pi a^3$. 13. 3·80 cu. ft.
14. 5·03 cu. ins. 15. 950 cu. ft. 16. 18 cu. ins.
17. 42·5 lbs. 18. 10 ozs. 19. $\frac{16}{3}a^3$.

EXAMPLES VIII a (p. 100)

1. 44·3; 41·5; 43·5. 2. Simpson $\frac{1}{3}$; Dufton I, 0·335; Dufton II, $\frac{1}{3}$; $\frac{1}{3}$.
3. 35·4; 38·5; 10, 2 per cent. 4. 52·7. 5. 1·81. 6. 3·909.

EXAMPLES VIII b (p. 103)

1. 330. 2. 370 cu. ins. 3. 278 ft. secs. 4. 5·9 mi. 8. 537,600 cu. ft.

EXAMPLES VIII c (p. 108)

1. $3\frac{3}{4}$. 2. 3 ins. 3. 29·3, 33·4 sq. ins. 4. $\frac{3k}{4}$.
5. $u+\frac{1}{2}p(t_1+t_2)+\frac{1}{3}q(t_1^2+t_1t_2+t_2^2)$; $u+\frac{1}{2}p(t_1+t_2)+\frac{1}{2}q(t_1^2+t_2^2)$.
6. $16t$, $\frac{64t}{3}$. 7. $\frac{ka}{2l}$. 8. 102.

REVISION PAPERS R. 6—11 (p. 109)

R. 6. 1. $\frac{3}{4}x^4-\frac{1}{4}x^2+c$; $-\dfrac{3}{x}+\dfrac{1}{x^2}+c$; $\frac{10}{9}x^{0\cdot9}+c$; $-\frac{1}{20}(2-5x)^4+c$.
2. $y=\frac{2}{3}x^3-\frac{3}{2}x^2+x$. 3. $x-y+1=0$, $x+y=3$; 2, 2. 4. 5602. 5. 2.

R. 7. 1. 4 ft. secs. 3. $-\frac{4}{3}$. 4. $\frac{1}{6}$; $\frac{52}{3}$; 4; $\dfrac{2\pi}{3}$; 0·636. 5. 440π.

R. 8. 1. 0·5075, 0·4615, 0·4145; 1·384. 3. 108, 108.
4. $y=\frac{1}{12}(2-x)^4+ax+b$. 5. 2; 3200 cm. gr.

R. 9. 1. $2+\dfrac{1}{x^2}$; $18x(3x^2+7)^2$; $1\cdot2x^{-0\cdot4}$; $\dfrac{2x}{(4-x^2)^2}$. 2. $\dfrac{5\pi}{6}$. 3. $\dfrac{k^3}{12}$.
4. 45°, 108° 26′. 5. 24.

R. 10. 1. If $f(x)=a+bx+cx^2$, $\frac{4}{3}\pi r^3$. 2. $\frac{4}{3}\pi r^3$. 3. $y^2=4ax$. 5. $\frac{1}{4}l^3$.

R. 11. 1. 18° 26′. 2. $\frac{9}{16}$. 3. $AN=3$. 4. $\dfrac{1}{4t^3}$; $\frac{1}{3}(a+1)^3$. 5. 5·06 ft. secs.

EXAMPLES IX a (p. 114)

1. 5·16 mi. 2. 296 ft. 3. $14\frac{2}{3}$ ft. 4. 38 ft. secs.; $82\frac{1}{2}$ ft.
$66\frac{2}{3}$ rad. secs. 6. $\dfrac{u}{1+ku}$ ft. secs. 7. 4·98 ft. secs. 8. 5·20 ft. secs.
9. 11·3 rad. secs. 10. $x^3=ay^2$.

EXAMPLES IX b (p. 120)

1. 27,400 ft. lbs. 2. 250 ft. lbs.; 80 ft. secs. 3. $\frac{4}{3}$ ft. lbs.
4. $\dfrac{p}{2l}(b-a)(b+a-2l)$ ft. lbs. 5. 420 ft. lbs. 6. $\frac{5}{8}$ ft. lbs. 7. 346 ft. lbs.
8. 117,000 ft. lbs. 9. $6\frac{1}{4}$ ft. lbs. 10. 80 ft. secs.
11. $2\frac{1}{2}$ ft. lbs.; 6·63 ft. secs.; falls 3·2 ft. 12. $\dfrac{E}{r}$. 13. $\frac{4}{15}$.
14. $\dfrac{rm}{r+m}$ mile-tons; $m-\dfrac{m^2}{2r}$ mile-tons. 15. 9 lb. secs.; 28·8 ft. secs.
16. 125 ft. secs. 17. 2230 ft.-tons. 18. $(100x-250)$ gr.; 2812·5 cm. gr.
19. 7500 ft. lbs.

EXAMPLES IX c (p. 125)

1. 3″, 2″. 2. (2·4, 0). 3. (1, 0·4). 4. $\dfrac{3r}{8}$ from base.
5. 1·5 ins. above base. 6. $(0, \frac{5}{8})$. 7. $(\frac{5}{8}, 0)$. 8. 48 cu. ins.; $\frac{13}{6}$ ins.
9. $\dfrac{99\pi}{2}$ cu. ins.; $\frac{37}{11}$ ins. 10. $\frac{1}{4}\pi^2(b-a)^2(b+a)$ cu. ins

EXAMPLES X a (p. 133)

1. $\frac{4}{3}\sqrt{3}=2\cdot31$ ft. 2. 135 lb. ft.^{2}. 3. $14\cdot3$ lb. ft.^{2}. 4. $\frac{1}{3}\sqrt{15}=1\cdot29$ ft.

5. $42\cdot2$ lb. ft.^{2}; $\frac{9}{2}\sqrt{5}=10\cdot1$ ins. 6. $\frac{1600\pi^2}{g}=494$ ft. lbs.

7. $\frac{7}{6}ma^2$. 8. $\frac{200\pi^2}{9g}=6\cdot86$ ft. lbs. 9. $\frac{3}{10}mr^2$. 10. $\frac{\sqrt{3}}{6}a=0\cdot29a$ in.

11. $\frac{3}{7}mb^2$. 12. $37\cdot5$ lb. ft.^{2}; $\frac{300\pi^2}{g}=92\cdot5$ ft. lbs.

EXAMPLES X b (p. 139)

1. $20\cdot8bh^2$ lbs. 2. $\frac{3}{4}h$ ft. 3. $17\cdot0$ ins. 4. 80 lb. ft.; 60 lbs.
5. 12000 lbs.; $5\frac{1}{3}$ ft. from top. 6. $40\cdot5$ lbs.; mid-point of median.
7. 247 tons; $10\frac{1}{3}$ ft.; no. 8. $103\cdot5$ kgs.; $6\cdot66$ cms. from vertex.
9. 350 lbs. 10. $1\frac{7}{8}$ ft. 11. The centroid. 12. $7\sqrt{2}=9\cdot90$ ins.

MISCELLANEOUS EXAMPLES M. 7—12 (p. 140)

M. 7. 1. $\frac{4\pi}{15}=0\cdot84$. 2. 6 sq. ins. 4. $165\frac{1}{3}$ lb. in.^{2}. 5. 6 rad. secs.; 5 rad.

M. 8. 1. 14 ins. 2. $(0, \frac{2}{3})$. 3. $\frac{5\pi^2}{9g}=0\cdot171$ ft. lb. 4. $3:5$. 5. $x^4=ay^3$.

M. 9. 1. max., $\frac{3-\sqrt{3}}{3}=0\cdot42$; min., $\frac{3+\sqrt{3}}{3}=1\cdot58$. 2. $2\frac{2}{3}$.
4. $96\cdot9$ tons; 151 lbs. per sec. 5. $27:107$.

M. 10. 1. $\frac{y_2(a+hx_2)-y_1(a+bx_1)}{b(1-n)}$; $1\cdot42$. 2. 18,200 cu. ft.
3. $2\cdot45(P_1V_1-P_2V_2)$ in. lbs.; $3\cdot92$ cu. ft.; 50,400 ft. lbs.
4. 28 kgs.; C. P. is $\frac{100}{3}$ cm. from FE and 10 cm. from EC.
5. $-\frac{1}{2}$ at $x=\frac{1}{3}$, min.; $\frac{2\sqrt{3}}{9}$ at $x=\frac{6-\sqrt{3}}{3}$, max.; $-\frac{2\sqrt{3}}{9}$ at $x=\frac{6+\sqrt{3}}{3}$, min.; least value, $-\frac{1}{2}$; yes.

M. 11. 2. $\frac{r}{5}\sqrt{10}$. 4. $3\cdot21$ tons.
5. $\frac{A}{Ca}\left[\sqrt{\left(\frac{2H_1}{g}\right)}-\sqrt{\left(\frac{2H_2}{g}\right)}\right]$ sec.; $30\cdot7$ min.

M. 12. 1. $972\pi=3050$ cu. ins. 2. 558 cu. ft.
4. 719 sq. ft.; $\frac{2x}{25}$ ft.; $4+\frac{3x}{100}$ ft.; $237\cdot5$ ft. tons. 5. 100 in. lbs.

PART II

EXAMPLES XI a (p. 150)

1. $15x^2+2$. 2. $20x^4-20x^3$. 3. $15x^5-5x^3$. 4. $2x+5$. 5. $6x^2-2x-1$.
6. $10x^4+20x^3+9x^2+22x+16$. 7. $3x^2+12x+11$. 8. $8x^3-3x^2+2$.
9. $\dfrac{1}{(x+2)^2}$. 10. $\dfrac{2-2x}{x^3}$. 11. $-\dfrac{29}{(5x-3)^2}$. 12. $\dfrac{2x}{(x^2+1)^2}$.
13. $\dfrac{(x+1)(x-3)}{(x-1)^2}$. 14. $\dfrac{2-2x}{(x+1)^3}$. 15. $\dfrac{3x^2-4x+19}{(3x-2)^2}$. 16. $10(2x+3)^4$.
17. $-24(4-3x)^7$. 18. $-12x(2-x^2)^5$. 19. $(x+2)(3x-4)$.
20. $\dfrac{1+x}{(1-x)^3}$. 21. $2(ax+b)(cx+d)(2acx+bc+ad)$. 22. $\dfrac{(5x-3)^2(42-5x)}{(x+2)^5}$.
23. $\dfrac{2(2x-3)(3-7x)}{(2x-3x^2)^2}$. 24. $2(3x^2+1)(x^3+x-5)$. 25. $\dfrac{1}{(x+2)^2}-\dfrac{1}{(x+1)^2}$.
26. $-\dfrac{4x}{(x^2+4)^3}$. 27. $\dfrac{4(x-1)}{(x+1)^3}$.

EXAMPLES XI b (p. 152)

1. $-\dfrac{3}{x^4}$, $2x+\dfrac{4}{x^3}$, $-\dfrac{4}{x^2}-\dfrac{15}{x^4}$, $-\dfrac{100}{x^{11}}$, $-\dfrac{2n}{x^{2n+1}}$.

2. $1{\cdot}5x^{\frac{1}{2}}$, $\frac{2}{3}x^{-\frac{1}{3}}$, $\frac{1}{3}x^{-\frac{2}{3}}$, $\frac{9}{2}x^{\frac{1}{2}}$, $\frac{5}{4}x^{\frac{3}{2}}$, $\dfrac{1}{q}x^{\frac{1}{q}-1}$, $\dfrac{2}{n}x^{\frac{2}{n}-1}$.

3. $-\frac{1}{2}x^{-\frac{3}{2}}$, $-3x^{-4}$, $-\frac{1}{3}x^{-\frac{4}{3}}$, 0, $-\frac{4}{3}x^{-\frac{4}{3}}$, $-\dfrac{1}{q}x^{-\frac{1}{q}-1}$.

4. $-2{\cdot}3x^{-3{\cdot}3}$, $-6x^{-3}$, $-\frac{3}{2}\sqrt{5}\,x^{-2{\cdot}5}$, $x^{-\frac{2}{3}}$, $-2x^{-\frac{3}{2}}$, $-3x^{-\frac{5}{2}}$, $-\dfrac{1}{\sqrt{(5-2x)}}$.

5. $3x^2+2+\dfrac{3}{x^2}+\dfrac{18}{x^4}$. 6. $3nx^{3n-1}+2nx^{2n-1}-3nx^{n-1}$. 7. $\dfrac{1}{2\sqrt{x}\,(1+\sqrt{x})^2}$.

8. $\frac{1}{2}(3-x)^{-1{\cdot}5}$. 9. $(x-1)(x^2-2x)^{-\frac{1}{2}}$. 10. $\dfrac{x}{(9-x^2)^{\frac{3}{2}}}$. 11. $3x^2(3x^3+8)^{-\frac{2}{3}}$.

12. $-(x^2-1)^{-1{\cdot}5}$. 13. $\dfrac{2x-3x^3}{\sqrt{(1-x^2)}}$. 14. $\frac{3}{2}(1-x)^{-\frac{3}{2}}(1+2x)^{-\frac{1}{2}}$.

15. $1+\dfrac{x}{\sqrt{(1+x^2)}}$ 16. $\dfrac{3[x+\sqrt{(1+x^2)}]^3}{\sqrt{(1+x^2)}}$. 17. $\dfrac{5x+9}{2(1+x)^2\sqrt{(2x^2-x-5)}}$.

18. $\dfrac{1}{2}\left(\dfrac{1}{\sqrt{(x+1)}}+\dfrac{1}{\sqrt{x}}\right)$. 19. $\mp x^{-\frac{3}{2}}$, $\mp x^{\frac{3}{2}}$, 1. 26. $0{\cdot}013$ sec.
27. Min., $\frac{27}{32}$; inflexion, 0. 28. $\frac{500}{121}$, $\frac{5}{4}$ ft. sec. 29. Max., $x=2$; min., $x=\frac{48}{19}$.
30. a^2. 31. 20 ft. from A. 32. $\frac{3}{2}\sqrt{2}=2{\cdot}12$ ft. 33. $\frac{6}{7}\sqrt{7}=2{\cdot}27$ m.

EXAMPLES XI c (p. 156)

1. $-\dfrac{x}{y}$. 2. $\dfrac{2x}{3y}$. 3. $-\dfrac{y^2}{x^2}$. 4. $\dfrac{y-x^2}{y^2-x}$. 5. $-\dfrac{2xy+y^2}{x^2+2xy}$. 6. $-\dfrac{3y}{2x}$.
7. $\dfrac{y+2}{3-x}$. 8. $-\sqrt{\left(\dfrac{y}{x}\right)}$. 9. $\dfrac{1}{t}$. 10. $\dfrac{t-1}{t}$. 11. $\dfrac{m(2-m^3)}{1-2m^3}$. 12. $\dfrac{2t}{t^2-1}$.

13. 4 in. sec. **14.** $41° 59'$ with horiz. **15.** 2·5 ft. sec. **17.** $x+y$.
18. $-\sqrt[3]{\left(\frac{y}{x}\right)}$. **19.** 4. **20.** 3 ft. sec. **21.** 4 in. min. **23.** 1·09 ft. sec.
25. $0·16l$ ft. sec. **26.** $1; -\frac{4}{3at^4}$. **27.** 3; 4·8 in. sec.

MISCELLANEOUS EXAMPLES M. 13—17 (p. 158)

M.13. **2.** $\sqrt{(625t^2-1200t+1600)}$; 0·96. **4.** 5·985 sq. ft. sec.
5. Least and min., $x=\frac{50}{8+\pi}$; greatest not max., $x=\frac{50}{\pi}$.

M.14. **2.** $\sqrt{(bc)}$. **3.** 1. **4.** 150π sq. ft. sec. **5.** $3\sqrt{2}$.

M.15. **1.** $12y=x$, $x+6y=1$, $x+5y+1=0$. **4.** $h=\frac{\lambda-\gamma}{\lambda-1}\cdot p$.

M.16. **1.** $p_2=0·53p_1$. **2.** $\theta^2=\frac{8c}{27(a+b)}$. **5.** 3·14 atmos.

M.17. **1.** length of chord. **2.** $\frac{8x+5x^2}{2\sqrt{(2+x)}}$; 0 at $x=0$, ∞ at $x=-2$; $x=-1·6$;
$123° 41'$. **3.** $\frac{2·4}{k}\sqrt{c}$. **4.** $-0·0024$; 0·7676. **5.** $-s$ ft. sec.^{2}.

EXAMPLES XII a (p. 169)

1. $3\cos(3x+4)$, $-3\sin 3x$, $-\cos(2-x)$. **2.** $4\sec^2 4(x-2)$, $\sin(a-x)$, $\frac{2\pi}{3}\cos\frac{2\pi}{3}(x+4)$. **3.** $2\sec 2x\tan 2x$, $-\sec(3-x)\tan(3-x)$, $4\sec^2(4x-3)$.

4. $-4\operatorname{cosec} 4x\cot 4x$, $\frac{2\pi}{a}\cos\frac{2\pi}{a}(x-b)$, $-3\operatorname{cosec}(3x-1)\cot(3x-1)$.

5. $-a\operatorname{cosec}^2 ax$, $\operatorname{cosec}^2(2-x)$, $-3\sec^2 3(2-x)$.

6. $\sin 2x$, $-2\sin 4x$, $2\tan(x-2)\sec^2(x-2)$.

7. $3\sin(6x-8)$, $-9\cos^2 3x\sin 3x$, $-2\operatorname{cosec}^2(x-2)\cot(x-2)$.

8. $2a\sec^2(ax-b)\tan(ax-b)$, $-6\operatorname{cosec}^3 2x\cot 2x$, $-\frac{1}{2}x^{-\frac{1}{2}}\operatorname{cosec}^2(x^{\frac{1}{2}})$.

9. $\sin x+x\cos x$, $2x\cos x-x^2\sin x$, $x\cos x$.

10. $\tan^2 x$, $\tan x+x\sec^2 x$, $\cos x\cos 2x-2\sin x\sin 2x$.

11. $\cos x\sin 2x+2\sin x\cos 2x$, $a\cos ax\cos bx-b\sin ax\sin bx$, $-p\sin px\cos qx-q\cos px\sin qx$.

12. $\cos^2 x$, $2\sec^2 2x\cos x-\tan 2x\sin x$, $2x\operatorname{cosec} x-x^2\operatorname{cosec} x\cot x$.

13. $\frac{x\cos x-\sin x}{x^2}$, $\frac{-2\cos x}{(1+\sin x)^2}$, $-\frac{\cos x}{(1+\sin x)^2}$.

14. $\cos x+\sin x$, $\frac{a}{x^2}\sin\frac{a}{x}$, $\frac{\cos 2x}{\sqrt{(\sin 2x)}}$.

15. $\sec^2\frac{x}{2}\tan\frac{x}{2}$, $\sec^2\frac{x}{2}\tan\frac{x}{2}$, $-\frac{1}{1+\sin 2x}$.

16. $\frac{\pi}{90}\cos(2x°)$, $-\frac{\pi}{180}\sin(2x°)$, $\frac{\pi}{360}\sec^2(\frac{1}{2}x°)$.

EXAMPLES XII b (p. 171)

1. $\frac{1}{2}\sin 2x + c$; $-\frac{1}{3}\cos 3x + c$.
2. $\frac{1}{2}\tan 2x + c$; $\frac{1}{2}\tan 2x - x + c$.
3. $-\frac{\cos(ax+bx)}{2(a+b)} - \frac{\cos(ax-bx)}{2(a-b)} + c$; $\frac{x}{2} + \frac{1}{4}\sin 2x + c$.
7. $5, -5$.
8. $\frac{2\sqrt{3}}{3}$ ft. sec.
9. $0{\cdot}0174$, $0{\cdot}0164$, $0{\cdot}0133$, $0{\cdot}0088$.
10. $-a\omega\sin(\omega t)$; $-a\omega^2\cos(\omega t)$.
15. $\frac{d\theta}{dt} = -\frac{v}{2a}\cos\theta\cot\theta$; $\frac{v}{2}\operatorname{cosec}\theta$; $-\frac{v}{2a\sqrt{2}}$; $\frac{v}{2}\sqrt{2}$.
16. $\frac{\pi\sqrt{2}}{180} = 0{\cdot}0247$ ft. sec.; $0{\cdot}0247$ ft. sec.

EXAMPLES XII c (p. 175)

1. $\frac{2}{\sqrt{(1-4x^2)}}$.
2. $-\frac{3}{\sqrt{(1-9x^2)}}$.
3. $\frac{4}{1+16x^2}$.
4. $\frac{1}{\sqrt{(9-x^2)}}$.
5. $\frac{1}{x\sqrt{(1-x^2)}} - \frac{1}{x^2}\sin^{-1}x$.
6. $2x\cos^{-1}x - \frac{x^2}{\sqrt{(1-x^2)}}$.
7. $-\frac{a}{x\sqrt{(x^2-a^2)}}$.
8. $\frac{a}{x\sqrt{(x^2-a^2)}}$.
9. $-\frac{a}{x^2+a^2}$.
10. 0.
11. $2x\tan^{-1}x$.
12. $\frac{1}{\sqrt{(1-x^2)}}$.
13. $\frac{1}{a}$.
14. 1.
15. $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$.
16. $\sin^{-1}\left(\frac{x}{3}\right) + c$.
17. $\frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + c$.
18. $\cos^{-1}x$.

REVISION PAPERS R. 12—17 (p. 176)

R. 12. 2. $\frac{8}{15}$.
3. $\frac{x-1}{2x\sqrt{x}}$; $\frac{n}{2}x^{n-1}(x^n+2)^{-\frac{1}{2}}$; $\frac{1}{\cos x - 1}$.
4. 3.
5. $\frac{1}{2}\sin x - \frac{1}{10}\sin 5x + c$; $\frac{1}{2}$.

R. 13. 2. 500 cu. ft.
3. $(\frac{3}{7}, 0)$.
4. $\frac{3x^2}{(4-x^3)^2}$; $1 - \frac{x}{\sqrt{(1+x^2)}}$; $3\sec^3 x\tan x$.
5. $-\frac{2}{3}x^{-\frac{4}{3}}$; $-\frac{3}{2}x^{\frac{4}{3}}$; 1.

R. 14. 1. $\frac{10}{3}\sqrt{\left(\frac{10}{3\pi}\right)} = 3{\cdot}43$ cu. in.
2. $\frac{4096\pi}{15} = 858$.
3. $\frac{x^2-4x-1}{(x^2+1)^2}$; $\frac{1}{x\sqrt{(4x^2-1)}}$; $-\frac{\pi}{60}\sin(3x^\circ)$.
4. $\pi\sqrt{3} = 5{\cdot}44$ in. sec.
5. 2220.

R. 15. 1. $\frac{2t}{t^2-1}$.
2. $-0{\cdot}05$.
3. $\frac{1}{2\sqrt{x}}\cos(\sqrt{x})$; $\frac{\cos x}{2\sqrt{(\sin x)}}$.
4. $\frac{9}{128}$.
5. $10(\sqrt{2}-1) = 4{\cdot}14$ in.

R. 16. 1. $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$; (i) $t = 0$ or $\sqrt[3]{2}$; (ii) $t = \infty$ or $\frac{1}{\sqrt[3]{2}}$.
2. $\frac{x^{n-1}}{\lfloor n-1}$; 0; $a^4\sin ax$.
3. $2a\omega\sin(\frac{1}{2}\omega t)$.
4. $\sqrt{(\frac{541}{48})} = 3{\cdot}36$ in.; $\frac{4}{3}\sqrt{3} = 2{\cdot}31$ in.
5. $\sqrt{\left(\frac{3b}{a}\right)}$.

R. 17. **1.** $\sec x\sqrt{(\sec 2x)}$. **2.** 0·084 per cent.
3. $\frac{1}{20}$; 0·89; $-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$. **4.** $29\frac{1}{8}$ in. lb. **5.** 4·95 sq. ft.

MISCELLANEOUS EXAMPLES M. 18—23 (p. 179)

M. 18. **1.** $\frac{52\pi}{9}=18{\cdot}1$ cm. sec. **2.** $2a\sin^2\theta\sin 2\theta$; 60°.
3. 34·7. **4.** $-\omega^2V$; $-n^2V$; $\frac{\omega^2}{n^2}$.

M. 19. **1.** 0·946; $-12{\cdot}65$ ft. sec.². **3.** 2·00101.
5. $a^2\omega\sec^2\theta$; $ah\omega\operatorname{cosec}^2\theta$ sq. ft. sec.

M. 20. **1.** $-\frac{b}{a}\cot\theta$. **2.** 60°. **3.** $-8{\cdot}41$. **5.** $\theta=\lambda$.

M. 21. **1.** $(-2-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$; $\frac{\pi}{4}\sqrt{(5-2\sqrt{2})}=1{\cdot}16$ ft. sec.; $\frac{\pi^2}{16}=0{\cdot}617$ ft. sec.².
2. 0·1 rad. sec. **3.** $\frac{1}{\pi}(\pi-2\theta+\sin 2\theta)$; $-2a^2(1-\cos 2\theta)$.
4. $ae>bc$. **5.** $52{\cdot}8\tan^2\theta$ ft. sec.

M. 22 **2.** 9·27 ft. **4.** $-\omega x\tan\theta$.
5. $2l\sin\alpha\sin(\beta-\theta)\operatorname{cosec}(\alpha+\beta)+c\sin\theta$; $\tan\theta=\frac{c\cot\alpha-(2l-c)\cot\beta}{2l}$.

M. 23. **1.** Lose 2·2 sec. per day. **2.** $1:\sqrt{3}$.

EXAMPLES XIII a (p. 188)

1. $2e^{2x}$. **2.** e^x-e^{-x}. **3.** e^x+e^{-x}. **4.** $2x\,.\,e^{x^2}$. **5.** $a\,.\,e^{ax+b}$.
6. $e^x(x+1)$. **7.** $e^{3x}(3\sin 2x+2\cos 2x)$. **8.** $-e^{-2x}(2\cos 3x+3\sin 3x)$.
9. $e^{ax}\operatorname{cosec}^2 bx\,(a\sin bx-b\cos bx)$. **10.** $\frac{1}{x}$. **11.** $\frac{1}{x-1}$. **12.** $\frac{3}{x}$.
13. $-\tan x$. **14.** $\sec x\operatorname{cosec} x$. **15.** $\frac{1+\cos x}{x+\sin x}$. **16.** $2x\log x+x$.
17. $\frac{2a}{x^2-a^2}$. **18.** $\operatorname{cosec} x$. **19.** $\sec x$. **20.** $2\sec x$. **21.** $\frac{1}{2}e^{2x}+c$.
22. $-\frac{1}{3}e^{-3x}+c$. **23.** $\frac{1}{a}e^{ax+b}+c$. **24.** $5\log x+c$. **25.** $-\frac{3}{5}\log(4-5x)+c$.
26. $-\log(\cos x)+c$. **27.** $\frac{1}{3}\log(\sin 3x)+c$. **28.** $\log(1+\sin x)+c$.
29. $\log\left(\frac{x+2}{x+3}\right)+c$. **30.** $1+\log x$; $x\log x-x+c$.
31. 2·92, 2·75, 2·59, 2·44, 2·3, 2·3. **32.** 0·7077. **36.** 3 or -5.

EXAMPLES XIII b (p. 193)

1. $\log x$. **2.** $3e^{2x}$. **3.** $2e^{-3x}$. **4.** Gradient at P equals y. **5.** $e-1$.
6. 1. **7.** Max. at (0, 1). **8.** $1-e^{-\frac{T}{\lambda}}$. **11.** 1, $-e^{-\frac{\pi}{2}}=-0{\cdot}208$.
12. 1; b. **13.** 78·4 per cent.
14. $0{\cdot}1<x<1$ more rapidly, $1<x<10$ less rapidly. **15.** $5e^2=36{\cdot}9$ ft. sec.

EXAMPLES XIII c (p. 196)

1. $10^x \log 10$; $\frac{1}{x \log 10}$; $-3^{-x}\log 3$; $\frac{2x}{(1+x^2)\log 10}$; $-\frac{\tan x}{\log 10}$; $\frac{2^x}{\log 10}\left(\log x \log 2+\frac{1}{x}\right)$.
2. $\frac{10^x}{\log 10}+c$; $\frac{3^{2x}}{\log 9}+c$; $\frac{a^{cx}}{c\log a}+b$.
3. 0·41; 39; 3·2.
6. ac^3e^{cx}; $\frac{2}{x^3}$.
11. $p=\frac{a}{a^2+b^2}$; $q=-\frac{b}{a^2+b^2}$.
12. $s=\frac{c}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right)$.
13. 21·1 tons sq. in.
14. $e^{0\cdot182x}$; 0·378; 5·89.

EXAMPLES XIII d (p. 202)

1. $\frac{u}{k}(e^{kt}-1)$ ft.
2. 30·5°.
3. 6·93; 0·76 sec.
4. 1710 ft. sec.
5. 0·354.
6. 1007 cu. cm.
9. 55·7°.
10. 2·75.
11. $\omega_1\left(\frac{\omega_2}{\omega_1}\right)^t$.
12. 31·6 min.
13. $2g$; $4g\left(4+\frac{1}{e^5}\right)=16{\cdot}03g$.
14. $\frac{b}{n}$; $n<0$.
15. $A=0$, $B=T$.
16. $C=C_0e^{-kt}$; $p=p_0e^{klt}$; $p \,.\, C^l=p_0 \,.\, C_0{}^l$.
17. $2\frac{1}{2}$; 2·69; 3·37 sq. in.
18. 333 ft. sec.
19. $-\frac{V}{R}$.
20. 135 ft. lb.

EXAMPLES XIV a (p. 208)

1. (i) $2ax\,dx$; (ii) $-4x\,dx$; (iii) $(6x-5)\,dx$; (iv) dx; (v) $\frac{1}{2}x^{-\frac{1}{2}}dx$; (vi) $-\frac{1}{2}x^{-\frac{3}{2}}dx$; (vii) $\left(1-\frac{1}{x^2}\right)dx$; (viii) $-\sin\theta\,d\theta$; (ix) $\sec\theta\tan\theta\,d\theta$; (x) $2\sec^2(2x+3)\,dx$; (xi) $\frac{dx}{1+x^2}$; (xii) $2\sec^2\theta\tan\theta\,d\theta$; (xiii) $3\cos(3x+5)\,dx$; (xiv) $-\operatorname{cosec}(2a+x)\cot(2a+x)\,dx$; (xv) $-\sec(a-x)\tan(a-x)\,dx$; (xvi) $\frac{dx}{x}$; (xvii) $\frac{a\,dx}{ax+b}$; (xviii) $\frac{1}{2}(x^{-\frac{1}{2}}-3x^{-\frac{5}{2}})\,dx$; (xix) $a\,.\,e^{ax}dx$; (xx) $-3e^{-3x}dx$.
2. $x+c$; x^2+3x+c; $2\sqrt{x}+c$; $-\frac{5}{2v^{0\cdot4}}+c$; $-2u^{-\frac{1}{2}}+c$; $c+\log(u+1)$; $x+\frac{3}{x}+c$; $c+\sec\theta$; $c-\cot\theta$.
3. 0·0087.
4. 0·10 per cent.
5. 0·7 per cent.
6. $(a+b+c)$ per cent.
7. $\frac{32\sqrt{3}}{9}=6{\cdot}16$ sq. in.
8. Height = diameter.
9. A and B acute, 2 per cent.
10. 3·4°.
12. 4·14 in. sec.
13. 0·5 per cent.

EXAMPLES XIV b (p. 210)

1. $\frac{1}{2}(4W+w)\tan\alpha$.
2. $W\tan\theta$.
3. $\frac{\sqrt{6}}{6}=0{\cdot}41$ lb.; $\frac{7\sqrt{6}}{18}=0{\cdot}95$ lb.
4. $W\sqrt{3}$.

EXAMPLES XIV c (p. 212)

[*Note:* the arbitrary constant is omitted.]

1. $\frac{1}{3}(x+4)^3$. 2. $-\frac{1}{4}(2-x)^4$. 3. $\frac{1}{3}(2x+3)^{\frac{3}{2}}$. 4. $-\dfrac{1}{4(2x+1)^2}$.

5. $\frac{1}{3}(2x+5)^{\frac{3}{2}}$. 6. $\dfrac{2}{3a}(ax+b)^{\frac{3}{2}}$. 7. $\frac{1}{2}\sin(2x-5)$. 8. $\cos(4-x)$.

9. $\dfrac{1}{n}\tan(nx+a)$. 10. $\frac{1}{3}\sin^3 x$. 11. $\frac{1}{3}\sin^3 x$. 12. $\frac{1}{2}\tan^2 x$.

13. $\frac{1}{2}\tan^2 x$. 14. $-\frac{1}{2}\cot^2 x$. 15. $-\dfrac{1}{a}e^{-ax}$. 16. $\dfrac{1}{a}e^{ax+b}$.

17. $\frac{1}{2}\log(2x-3)$. 18. $\dfrac{1}{a}\log(ax+b)$. 19. $-\log(2-x)$. 20. $\frac{5}{7}x^{1\cdot 4}$.

21. $-0\cdot 6055$. 22. $\frac{5}{3}x^{0\cdot 6}$. 23. $\frac{1}{3}(x^2+2x+3)^{\frac{3}{2}}$.

24. $\dfrac{1}{2(1+2\cos x)}$. 25. $-\frac{1}{2}e^{-x^2}$.

EXAMPLES XIV d (p. 215)

[*Note:* the arbitrary constant is omitted.]

1. $-\frac{1}{3}(3-x^2)^{\frac{3}{2}}$. 2. $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$. 3. $-2\sqrt{(1-x)}$. 4. $-\sqrt{(1-x^2)}$.
5. $\frac{2}{3}\sqrt{(a^2+x^3)}$. 6. $\frac{2}{3}\pi\sqrt{(y^2+a^2)^3}$. 7. $0\cdot 828$. 8. $0\cdot 490$.

9. $\frac{1}{3}\log(3+t^3)$. 10. $\dfrac{3}{2a}\log(b^2+a^2x^2)$. 11. $-\log(3-x)$.

12. $-\dfrac{1}{b}\log(a-bx)$. 13. $\log(\sin x)$. 14. $\log(x^3-x^2+1)$.

15. $\frac{1}{2}(\log x)^2$. 16. $\frac{1}{6}(1+x^4)^{\frac{3}{2}}$. 17. $\frac{1}{4}\sin^4 x$. 18. $0\cdot 0841$. 19. $\frac{1}{6}\tan^6\theta$.

EXAMPLES XIV e (p. 219)

[*Note:* the arbitrary constant is omitted.]

1. $\frac{1}{2}x+\frac{1}{4}\sin 2x$. 2. $\frac{1}{2}x+\frac{1}{12}\sin 6x$. 3. $-\dfrac{\cos(ax+bx)}{2(a+b)}-\dfrac{\cos(ax-bx)}{2(a-b)}$.
4. $\tan x - x$. 5. $\cos(1-x)$. 6. $0\cdot 123$. 7. $\frac{1}{2}T$.
8. $\frac{1}{12}(\cos 3x - 9\cos x)$. 9. $\dfrac{\pi}{32}$. 10. $\sin^{-1}\left(\dfrac{x}{4}\right)$. 11. π.

12. $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)$. 13. $\dfrac{\pi}{4a}$. 14. $\tan^{-1}(x+3)$.

15. $-\sin^{-1}\left(\dfrac{1-x}{2}\right)$. 16. $\dfrac{7\sqrt{3}}{6}\sin^{-1}(x\sqrt{\tfrac{3}{7}})+\dfrac{x}{2}\sqrt{(7-3x^2)}$. 17. $3\cdot 55$.

18. $-\sin^{-1}\left(\dfrac{a-2x}{a}\right)$. 19. $\sin^{-1}(x-3)$. 20. $-x-\dfrac{1}{a}\cot(ax+b)$.

21. $\dfrac{3\pi}{16}$. 24. $\dfrac{\pi}{16}$. 25. $\dfrac{\pi}{4}$. 26. k; $\dfrac{2k}{\pi}$.

EXAMPLES XIV f (p. 223)

[*Note:* the arbitrary constant is omitted.]

1. $x-\log(1+x)$. 2. $x-\log(x+3)$. 3. $\frac{1}{2}x^2-2x+7\log(x+2)$.

4. $\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$. 5. $\frac{1}{12}\log\left(\frac{2x-3}{2x+3}\right)$. 6. $\frac{1}{2}\log(x^2-4)$.

7. $\frac{1}{2}\tan^{-1}(2x)$. 8. $\frac{3}{8}\log(4x^2+1)+\tan^{-1}(2x)$.

9. $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}\log(x^2+16)$. 10. $\frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right)$.

11. $\frac{5}{2}\log(x^2-2x+10)+\frac{5}{3}\tan^{-1}\left(\frac{x-1}{3}\right)$.

12. $\frac{1}{2}\log[(x+a)^2+b^2]+\frac{1-a}{b}\tan^{-1}\left(\frac{x+a}{b}\right)$. 13. $\log\left(\frac{x+1}{x+2}\right)$.

14. $4\log(x-3)-3\log(x-2)$. 15. $-\log[(1-x)^2(2+x)^3]$.

16. $-\log[(5+x)^2(2-x)]$. 17. $\frac{1}{x}+\log\left(1-\frac{1}{x}\right)$.

18. $\log\left(\frac{x+2}{x-1}\right)-\frac{1}{x-1}$. 19. $\frac{2}{3}x^3-\frac{1}{2}x^2+2x-3\log(x+2)$.

20. $3\log(x-2)-\frac{3}{2}\log(x^2+8x+17)+12\tan^{-1}(x+4)$.

21. $\frac{1}{11}\log(x-2)-\frac{1}{22}\log(x^2+3x+1)-\frac{7}{22\sqrt{5}}\log\frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}}$.

22. $2\log(x-2)-\frac{3}{2}\log(x-1)-\frac{1}{2}\log(x-3)$.

23. $\log(x+1)+\frac{2}{x+1}-\frac{1}{2(x+1)^2}$.

24. $\log x-\frac{1}{2}\log(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$. 25. 0·307. 26. $\frac{\pi\sqrt{3}}{9}$.

27. 8·27. 28. $\log\left(\frac{2a}{a+1}\right)+\frac{1-a}{2(1+a)}$; 0·193.

29. $y=x+1-2\log(x+2)$. 30. $\frac{1}{2kV}\log 3$.

EXAMPLES XIV g (p. 226)

[*Note:* the arbitrary constant is omitted.]

1. $\frac{1}{4}(2x^2\log x-x^2)$. 2. $e^x(x-1)$. 3. $\frac{1}{9}(3x^3\log x-x^3)$.

4. $\sin x-x\cos x$. 5. $\frac{x(1+x)^{11}}{11}-\frac{(1+x)^{12}}{132}$. 6. $\frac{1}{9}(3x\sin 3x+\cos 3x)$.

7. $x\tan x+\log\cos x$. 8. $\frac{1}{2}(1+x^2)\tan^{-1}x-\frac{1}{2}x$. 9. $\frac{1}{4}(\sin 2x-2x\cos 2x)$.

10. $x\sin^{-1}x+\sqrt{(1-x^2)}$. 11. $(x^2-2)\sin x+2x\cos x$. 12. $-\frac{1+2\log x}{4x^2}$.

13. $\frac{1}{2}e^x(\sin x-\cos x)$. 14. $\frac{1}{2}e^x(\sin x+\cos x)$. 15. $-\cos x+\frac{2}{3}\cos^3x-\frac{1}{5}\cos^5x$.

16. $\frac{x^{n+1}}{(n+1)^2}[(n+1)\log x-1]$. 17. $-\frac{1}{13}e^{-2x}(2\sin 3x+3\cos 3x)$.

18. $-\frac{(1-x)^{21}(231x^2+21x+1)}{5313}$.

20. $-x^5\cos x+5x^4\sin x+20x^3\cos x-60x^2\sin x-120x\cos x+120\sin x$.

21. $\frac{35\pi}{256}$. 23. Integral $=-x^ne^{-x}+n\int x^{n-1}e^{-x}dx$; $\underline{|10}$. 24. $\frac{3\pi}{512}$; $\frac{16}{1155}$.

EXAMPLES XIV h (p. 231)

[*Note:* the arbitrary constant is omitted.]

1. $2\log\left(\tan\frac{\theta}{4}\right)$. 2. $2\log\left[\tan\frac{\pi+\theta}{4}\right]$. 3. $\log\tan\theta$.

4. $\frac{1}{\sqrt{2}}\log\left[\tan\frac{4x+\pi}{8}\right]$. 5. $\frac{1}{2}\tan^{-1}\left[\frac{1}{2}\tan\frac{x}{2}\right]$. 6. $\frac{1}{4}\log\left(\dfrac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right)$.

7. $\log\left(\frac{\sqrt{5}+2}{\sqrt{2}+1}\right)$. 8. $\frac{1}{2}[\sqrt{2}+\log(1+\sqrt{2})]$. 9. $\frac{\pi}{\sqrt{5}}$.

10. $\frac{x}{4}(1+x^2)^{\frac{3}{2}}+\frac{3x}{8}\sqrt{(1+x^2)}+\frac{3}{8}\log[x+\sqrt{(1+x^2)}]$.

11. $\frac{x}{2}\sqrt{(x^2-16)}-8\log[x+\sqrt{(x^2-16)}]$.

12. $\frac{1}{4}(2x-3)\sqrt{(x^2-3x-10)}-\frac{49}{8}\log[x-\frac{3}{2}+\sqrt{(x^2-3x-10)}]$.

13. $\frac{8}{15}$. 14. $\frac{5\pi}{32}$. 15. $\frac{2}{143}$. 16. $\frac{5\pi}{256}$. 17. $\frac{5\pi}{32}$. 18. $\frac{16}{1155}$. 19. $\frac{1}{280}$.

20. $\frac{\pi a^4}{16}$. 21. $\frac{\pi}{2}$. 22. $\frac{\pi}{8}$. 23. $\frac{3\pi a^2}{16}$. 24. $\frac{3\pi}{16}$. 25. $\frac{3\pi}{8}$.

EXAMPLES XIV i (p. 232)

[*Note:* the arbitrary constant is omitted.]

1. $-\frac{x^{-0\cdot37}}{0\cdot37}$. 2. $\frac{3}{4}(a+x)^{\frac{4}{3}}$. 3. $\tan^{-1}x$. 4. $\frac{3}{20}(1+2x)^{\frac{5}{3}}-\frac{3}{8}(1+2x)^{\frac{2}{3}}$.

5. $-\sqrt{(a^2-x^2)}$. 6. $-\log(c-x)$. 7. $-\frac{1}{6}\log(1-2x^3)$.

8. $\frac{1}{2}\log(x^2+a^2)+\tan^{-1}\left(\frac{x}{a}\right)$. 9. $-\frac{1}{b}\log(a+b\cos x)$. 10. $-\frac{1}{b(a+bx)}$.

11. $\frac{1}{9}\left[\frac{1}{1-3x}+\log(1-3x)\right]$. 12. $\log\left(1+\frac{1}{x}\right)-\frac{1}{x}$.

13. $\frac{2}{27}(2+3x)^{\frac{3}{2}}-\frac{4}{9}(2+3x)^{\frac{1}{2}}$. 14. $\frac{2}{3}(1+\log x)^{\frac{3}{2}}$. 15. $\frac{\log x}{\log a}$.

16. $\frac{x}{8}-\frac{\sin 4x}{32}$. 17. $\log(\sin x)$. 18. $\frac{1}{3}\log\left(\tan\frac{3x}{2}\right)$. 19. $-\frac{1}{3}\cot 3x$.

20. $\frac{1}{2}(x^2-8x-20)+12\log(x+2)+\frac{8}{x+2}$. 21. $\frac{\sin(ax-bx)}{2(a-b)}-\frac{\sin(ax+bx)}{2(a+b)}$.

22. $\frac{5x^2}{2}+15x-6\log(x-1)+41\log(x-2)$. 23. $\frac{3}{4}\log(x+3)+\frac{1}{4}\log(x-1)$.

24. $\frac{1}{4}\log\frac{1+x^2}{(1+x)^2}+\frac{1}{2}\tan^{-1}x$. 25. $\frac{1}{2}x-\frac{1}{12}\sin 6x$. 26. $\frac{x^2}{4}(2\log x-1)$.

27. $x\cos^{-1}x-\sqrt{(1-x^2)}$. 28. $\log\left(\tan\frac{\pi+2x}{4}\right)$. 29. $\frac{1}{8}\sin^4 2x$.

30. $\sqrt{(x^2-a^2)}+a\sin^{-1}\left(\frac{a}{x}\right)$. 31. $\frac{3}{32}(3-2x)^{\frac{8}{3}}-\frac{9}{20}(3-2x)^{\frac{5}{3}}$.

32. $\frac{1}{5}x^{\frac{5}{2}}-5x^{\frac{1}{2}}$. 33. $\log x[\log(a\sqrt{x})]$. 34. $\log(1+\sin x)$.
35. $e^x(x^2-2x+2)$. 36. $-\cos(\log x)$. 37. $(\log x)^2$. 38. $\log(1+\log x)$.
39. $\frac{1}{2}(\tan^{-1}x)^2$. 40. $2\sqrt{(e^x+4)}-2x+4\log[\sqrt{(e^x+4)}-2]$.
41. $\dfrac{1}{a}\tan(ax+b)-x$. 42. $\dfrac{7^x}{\log 7}$. 43. $x\log_2\left(\dfrac{x}{e}\right)$. 44. $\sin^{-1}x+\sqrt{(1-x^2)}$.
45. $\frac{2}{3}(x-1)^{\frac{3}{2}}+2(x-1)^{\frac{1}{2}}$. 46. $\frac{1}{4}\tan^4 x$. 47. $-\frac{1}{5}e^{-x}(\sin 2x+2\cos 2x)$.
48. $\frac{1}{2}\tan^2 x+\log\cos x$. 49. $\frac{1}{2}\tan^{-1}(2\tan x)$.
50. $\dfrac{1}{a}\{\log x-\log[a+\sqrt{(a^2-x^2)}]\}$. 51. $\dfrac{\pi}{4}-\frac{2}{3}$. 52. $\frac{1}{2}$. 53. $\frac{1}{60}$. 54. $\frac{16}{35}$.
55. $\dfrac{\pi}{2}$. 56. $\dfrac{(4-\pi)\sqrt{2}}{8}$. 57. $\frac{2}{27}$. 58. 1. 59. $\dfrac{35\pi}{256}$. 60. 1.
61. $\log\frac{32}{27}$. 62. $-\frac{1}{4}+\frac{1}{2}\log 2$. 63. $\dfrac{\pi}{8}$. 67. $\dfrac{\pi}{8}$. 68. $\frac{8}{15}$.

EXAMPLES XV a (p. 237)

1. $\phi=\theta,\ p=a\sin^2\theta$; $\phi=\pi-\theta,\ p=a$; $\phi=\dfrac{3\pi}{4}-\dfrac{\theta}{2},\ p=\dfrac{a}{2}\sec\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$; $\phi=a,\ p=r\sin a$.
3. $p^2=ar$. 4. $r^{n+1}=a^n p$. 5. $\theta=\dfrac{\pi}{12},\ r=\dfrac{a}{2}\sqrt[4]{12}$. 6. $r=a$.

EXAMPLES XV b (p. 239)

1. $\frac{1}{4}\pi a^2$. 2. $\dfrac{3\pi a^2}{2}$. 3. $\dfrac{\pi}{4}<\theta<\dfrac{3\pi}{4}$; $\frac{1}{2}a^2$. 5. $\frac{1}{2}c^2\log 3$.
6. Sum of areas of loops$=3\pi$; $r=2\cos\theta-1$ for $-\dfrac{2\pi}{3}<\theta<\dfrac{2\pi}{3}$; $\theta=\dfrac{2\pi}{3}$ to $\theta=\dfrac{4\pi}{3}$ for $r=1+2\cos\theta$. 8. $\frac{1}{2}ab\tan^{-1}\left(\dfrac{a\tan a}{b}\right)$.

EXAMPLES XV c (p. 241)

1. $\dfrac{2r}{\pi}$ from centre. 2. $\dfrac{4r}{3\pi}$ from centre. 5. $\dfrac{\pi a}{4\sqrt{2}}$. 6. $\dfrac{4a}{5}$ from origin.

EXAMPLES XV d (p. 244)

1. $\frac{1}{27}(13\sqrt{13}-8)$. 2. $4a$. 3. $8a$. 4. $\frac{61}{27}$. 5. $6a$.
6. $\dfrac{c}{2}\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)$. 8. $h+\frac{2}{3}a^2h^3$. 9. $a\sec\gamma\,(e^{\beta\cot\gamma}-e^{a\cot\gamma})$.

EXAMPLES XV e (p. 248)

1. $\dfrac{13\sqrt{13}}{6}$. 2. $\dfrac{1}{2a}$; $\dfrac{1}{2a}$. 3. $\dfrac{(x^2+y^2)^{\frac{3}{2}}}{2c^2}$; $\dfrac{y^2}{c}$. 4. $r\operatorname{cosec}a$; $\dfrac{4a}{3}\cos\dfrac{\theta}{2}$.
6. $4a\cos\dfrac{\theta}{2}$. 7. $3a\sin t\cos t$. 12. $\dfrac{1}{ab}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}$.

EXAMPLES XV f (p. 252)

1. πrl. 2. $\frac{8\pi a^2}{3}\left[\left(1+\frac{x}{a}\right)^{\frac{3}{2}}-1\right]$. 3. $\frac{1225\pi}{4}$ sq. in.

4. $\frac{12\pi a^2}{5}$. 5. $\frac{32\pi a^2}{5}$. 6. $2\pi a^2(2-\sqrt{2})$.

7. $\frac{\pi c^2}{4}\left(e^{\frac{2a}{c}}-e^{-\frac{2a}{c}}+\frac{4a}{c}\right)$; $\pi c\left[a\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)-c\left(e^{\frac{a}{c}}+e^{-\frac{a}{c}}\right)+2c\right]$.

8. Area of surface formed by revolution about $y=x\tan\alpha$. 9. $\frac{5\pi^2}{4}$ sq. in.

10. $\frac{1}{3}\pi r^2h$; $\pi r\sqrt{(r^2+h^2)}$. 11. $\frac{2r}{\pi}$ from centre; $\pi(\pi+1)$ sq. in.

12. $\frac{r\sin\alpha}{\alpha}$ from centre; $4\pi r^2(\sin\alpha-\alpha\cos\alpha)$. 13. $\frac{4r}{3\pi}$ from centre.

14. 1 ft. 15. $16\pi(\pi-1)$ sq. in.

EXAMPLES XV g (p. 256)

1. In Fig. 176, ends of chain are on OX.

11. $\sqrt{(4ag)}$; $p=4a$, $n=\sqrt{\left(\frac{g}{4a}\right)}$, $\epsilon=0$; $\pi\sqrt{\left(\frac{a}{g}\right)}$. 12. $2m\cos\psi$.

REVISION PAPERS R. 18–24 (p. 257)

R. 18. 1. $-3\sin(6x+4)$; $\frac{2}{\sqrt{(7-12x-4x^2)}}$. 3. $-\frac{x}{y}$; $-\sqrt{\left(\frac{y}{x}\right)}$; $-\frac{y}{x}$.

4. $a=\frac{RE}{R^2+L^2p^2}$, $b=\frac{LpE}{R^2+L^2p^2}$. 5. 22·4°.

R. 19. 1. $-\frac{a}{x^2+a^2}$; $\frac{1}{2}\operatorname{cosec}^{\frac{1}{2}}(1-x)\cot(1-x)$; $\frac{5\pi a^6}{32}$.

2. $x\sec\phi+y\operatorname{cosec}\phi=a$; $\frac{3a}{2}\sin 2\phi$. 4. $\frac{b\delta b-a\delta a}{c}$. 5. $\left(\frac{\pi^2+4}{8\pi}, 0\right)$.

R. 20. 1. $\frac{1}{x-1}$; $-4\operatorname{cosec}4x$; $\frac{1}{2}\log\frac{5}{3}$; 1. 2. $\frac{dV}{dt}=-kx$ where $V=\frac{1}{3}\pi x^3\tan^2\alpha$.

3. 315 ft. 4. 0·1823. 5. $\frac{12400}{\sqrt{3}}$ lb. = 3·20 tons.

R. 21. 1. $-\frac{1}{2}x^{-\frac{3}{2}}dx$; $ae^{ax}dx$; $\frac{b\,dx}{bx-a}$; $\sec^2x\,dx$.

3. $\frac{2}{e}$. 5. $\frac{v_0}{k}(1-e^{-kt})$ ft.; 3·92 sec.

R. 22. 1. $\frac{4}{5}x^5-\frac{20}{3}x^3+25x+c$; $\log(3x^2-5x+7)+c$; $\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+c$; $\frac{1}{16}(\sin 8\theta+4\sin 2\theta)+c$.

2. £122·1. 3. 0·7 per cent. 4. 164 ft.; 1·41 sec.

R. 23. **1.** $\frac{\pi}{2}$; $\log 3 - 1 = \cdot 0896$. **2.** $(p-x)(q-x)$.

3. $2e^{-\frac{1}{2}t}(6\cos 3t - \sin 3t)$; $-e^{-\frac{\pi}{6}} = -0{\cdot}592$. **5.** 1·36 tons.

R. 24. **1.** $\frac{1}{k(b-a)}\log\frac{a(b-x)}{b(a-x)}$. **2.** 92·1 per cent.

3. $\frac{1}{20}\log(x-2)+\frac{1}{5}\log(x+3)-\frac{1}{4}\log(x+2)+c$; π.

4. $\frac{a^2}{2}\log\left(\tan\frac{5\pi}{12}\right)$. **5.** $\frac{1}{2}a^2$.

MISCELLANEOUS EXAMPLES M 24—31 (p. 260).

M. 24. **2.** $3b \geqslant a^2$. **3.** $\frac{16\pi a^3}{5}$. **4.** $\theta=90°$, $r=1$, min.;

$\theta=54° 44$ or $125° 16'$, $r=\frac{4}{9}\sqrt{6}=1{\cdot}63$, max. **5.** $\frac{13\pi}{32}$.

M. 25. **1.** $e-\frac{3}{e}$. **2.** $\frac{1}{3}\pi a^2 b$.

3. $x^{\frac{1}{x}-2}(1-\log x)$; $(-1)^{n-1}\lfloor n-1 \left(\frac{1}{(x+1)^n}+\frac{1}{(x+3)^n}\right)$.

4. max. 8, min. 0. **5.** $r^2\cos 2\theta = a^2$.

M. 26. **1.** $83\frac{1}{3}$ min. **2.** 66·5 lb. **4.** $\frac{1}{e}$. **5.** $29\frac{1}{6}$ cu. in.

M. 27. **2.** 4, 1. **3.** $a=\frac{1}{2}$, $b=2$; 0, $2+\sqrt{2}$. **4.** $-bn$, $-am$; larger force; 7071 men. **5.** $A<9B$; $A=\frac{3}{2}$, $B=\frac{1}{2}$; $\sqrt{2}$.

M. 28. **1.** $\frac{1}{3}$. **2.** $\frac{\pi s_1^3}{6a}$. **4.** $y=\frac{gx^2}{3200}$; $\frac{1600}{g}[2\sqrt{3}+\log(2+\sqrt{3})]=239$ ft.

5. 8350 ft. lb.

M. 29. **2.** $\frac{4\pi^3 a^2}{3}$. **3.** 6·84 in above vertex; 8·4 lb. **4.** $\frac{a^2}{2}\left(e-\frac{1}{e}\right)$.

M. 30. **1.** $5\cot\frac{\theta}{2}$; $-\frac{5}{2}\operatorname{cosec}^2\frac{\theta}{2}\,\delta\theta$; 6·6 ft. **3.** $b-2a\cos\theta$;

$a(b-2a\cos\theta)\,\delta\theta$; $\frac{1}{2}\pi ab-2a^2$. **4.** 330 lb. **5.** $2\pi r^2(\pi-2)$.

M. 31. **3.** $\frac{a(1+e^{-b\pi})}{1+b^2}$. **4.** $\frac{3}{2}\sqrt[3]{(ax^2)}$.

EXAMPLES XVI a (p. 270)

1. x^2+y^2; $2i$; $\frac{1}{2}(i\sqrt{3}-1)$; -1. **2.** $\pm 2i$; $-4\pm 3i$. **4.** $a=3$, $b=5$.

5. $r=5$, $\theta=2n\pi+0{\cdot}927$ or $r=-5$, $\theta=(2n+1)\pi+0{\cdot}927$. **6.** -1, $1\pm i$.

13. $\cos 5\theta + i\sin 5\theta$; $2\cos\frac{\theta}{2}$. **16.** 0, 1, 0. **20.** $\cosh n\theta - \sinh n\theta = e^{-n\theta}$.

22. $\sec\theta$; $\sin\theta$. **24.** $\pm 1{\cdot}32$.

EXAMPLES XVI b (p. 273)

3. $\dfrac{1}{\sqrt{(a^2+x^2)}}$. 4. $\dfrac{a}{a^2-x^2}$. 6. $\dfrac{1}{n}\cosh(nx)$; $\dfrac{1}{n}\sinh(nx)$.

8. $\dfrac{1}{n}\tanh(nx)$; $-\dfrac{1}{n}\coth(nx)$. 9. $\frac{1}{4}\sinh 2\theta$.

11. $\frac{1}{3}\sinh^{-1}(3x)$; $\cosh^{-1}\left(\dfrac{x}{2}\right)$. 12. $\cosh^{-1}\left(\dfrac{x+a}{b}\right)$; $\cosh^{-1}\left(\dfrac{x+5}{3}\right)$.

13. $\frac{1}{2}x\sqrt{(1+x^2)}-\frac{1}{2}\sinh^{-1}x$. 14. $\frac{1}{2}\cosh(ax+bx)-\frac{1}{2}\cosh(ax-bx)$; $\dfrac{\sinh(ax+bx)}{2(a+b)}-\dfrac{\sinh(ax-bx)}{2(a-b)}$. 16. $s=c\sinh\left(\dfrac{x}{c}\right)$. 18. $c\cosh^2\left(\dfrac{x}{c}\right)$.

19. $x\cosh x-\sinh x$; $\frac{1}{4}e^{2x}-\frac{1}{2}x$. 20. $\log\left(\tanh\dfrac{x}{2}\right)$; $x\sinh^{-1}x-\sqrt{(1+x^2)}$.

22. 0·56; 0·446; 55·7. 23. 1·407; 0·632. 24. 20 ft. per sec.; 0·34 sec.; 1·8 ft.

EXAMPLES XVII a (p. 279)

10. $1+x-\dfrac{x^3}{3}-\dfrac{x^4}{6}-\dfrac{x^5}{30}$. 11. $\log 2+\dfrac{x}{2}+\dfrac{x^2}{8}$. 12. $1+x+\frac{1}{2}x^2+\frac{1}{2}x^3$.

13. $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4$.

EXAMPLES XVII b (p. 281)

1. $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\dots$. 2. $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\dots$.

3. $\log\dfrac{1+x}{1-x}=2(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\dots)$. 4. $\dfrac{2}{(1-x)^3}$. 5. $(1+x)\log(1+x)-x$.

6. $(1-x^2)^{-\frac{3}{2}}$. 7. $\cos x-x\sin x$. 8. $\dfrac{\sinh x}{2x}+\frac{1}{2}\cosh x$.

9. $\frac{1}{2}\tan^{-1}x+\dfrac{x}{2(1+x^2)}$. 11. $\dfrac{1}{2}\left\{\dfrac{(2x)^2}{\lfloor 2}-\dfrac{(2x)^4}{\lfloor 4}+\dfrac{(2x)^6}{\lfloor 6}-\dots\right\}$.

12. $\dfrac{1}{2}\left\{2x-\dfrac{(2x)^3}{\lfloor 3}+\dfrac{(2x)^5}{\lfloor 5}-\dots\right\}$. 14. $x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\dfrac{x^7}{7}+\dots$.

15. $x-\frac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}-\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\dots$.

EXAMPLES XVII d (p. 291)

1. $\frac{1}{6}$. 2. 1. 3. 2. 4. $\frac{4}{5}$. 5. $-\frac{1}{3}$. 6. 0. 7. $\frac{1}{3}$. 8. $\frac{1}{2}$.

9. 0. 10. $a_0=1$; $a_1=-2n^2$. 11. $a_0=2n$; $a_1=\frac{4}{3}n(1-n^2)$.

MISCELLANEOUS EXAMPLES M. 32—35 (p. 291)

M. 32. 1. $-\dfrac{a}{(\theta-a)^2}$; $\dfrac{x^2}{2}-\dfrac{x^4}{12}$. 3. $\dfrac{a}{3}$. 4. πa. 5. point of inflexion.

M. 33. 3. $y=a\,.\,e^{-\frac{x}{y}}$. 4. $4a^3\left(1-\dfrac{2}{\pi}\right)$; $\dfrac{a(\pi+2)}{16}$.

M. 34. 1. $\frac{4}{5}$ ft. 2. $\frac{1}{2}\pi^2-\frac{1}{4}\pi$ sq. in. 3. $(\sqrt[4]{\frac{5}{3}}, 3\frac{1}{3})$.

M. 35. 1. $\angle ACP$ obtuse, 50 ft. min.; $\angle ACP$ acute, 62·8 ft. min.

4. $\sqrt{\frac{128}{35}}$. 5. $\sqrt{\{\frac{1}{2}(a^2+p^2)\}}$.